Mixed elliptic motives

Alexander Goncharov

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1 Introduction

Summery. Let E be an elliptic curve over an arbitrary field k and $\mathcal{H}$ the motive $H^1(E)(1)$. We define complexes $B(E;n+2)^{\bullet}$ and conjecture that they are quasi-isomorphic to $RHom_{\mathcal{MM}_k}(\mathbb{Q}, Sym^n\mathcal{H}(1))$. If k is a number field this together with Beilinson's conjecture on regulators leads to a precise conjecture expressing the special values $L(Sym^n E, n+1)$ via the classical Eisenstein-Kronecker series. It can be considered as an elliptic analog of Zagier's conjecture.

We give a simple motivic interpretation of the elliptic polylogarithms and show how it together with the motivic formalism implies that the complexes $B(E;n+2)^{\bullet}$ should map naturally to $RHom_{\mathcal{MM}_k}(\mathbb{Q}, Sym^n\mathcal{H}(1))$. When E

degenerates to the nodal curve our complexes lead to the motivic complexes from [G1-2] reflecting the properties of the classical polylogarithms.

The complex $B(E;3)^{\bullet}$ was constructed in [GL]. The groups similar to $H^1 B(E;n+2)^{\bullet}$ were also discussed in [W], [W2], where it was conjectured that they inject into $Ext^1_{\mathcal{MM}_k}(\mathbb{Q}, Sym^n \mathcal{H}(1))$.

We formulate several conjectures about the structure of the motivic Galois group of the category of mixed elliptic motives generalizing some conjectures about the mixed Tate motives ([G1-2]). In the end we define the generalized Eisenstein-Kronecker series which should be related to $L(Sym^n E, n+m)$. For $n=1, m=2$ this was conjectured in [D3] and proved in [G3].

1. The trichotomy: special values of L-functions, motivic complexes and motivic Galois groups. Let E be an elliptic curve over a number field k and $L(Sym^n E, s)$ the L-function of the n-th symmetric power of $h^1(E)$. The seminal Beilinson conjecture relates the special values of the L-function of a motive X over a number field, considered up to a $\mathbb{Q}^*$-factor, with the volume of the image of certain pieces of the algebraic K-theory of X under the regulator maps. However for symmetric powers of elliptic curves one should be able to say more about the special values.

It is natural to adress the problem in the language of motives. Let $\mathcal{MM}_k$ (resp. $\mathcal{MM}_X$) be the (hypothetical) abelian $\mathbb{Q}$-category of all mixed motives over a field k (resp. all mixed motivic sheaves over a regular scheme X, [Be]). Let $\mathbb{Q}(-1) := h^2(P^1)$ be the Tate motive, $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ for any integer n and $M(n) := M \otimes \mathbb{Q}(n)$.

Let E be an elliptic curve over a field k. Then $h^1(E)$ is a pure motive of weight 1. The cup product defines an isomorphism $\Lambda^2 h^1(E) \longrightarrow \mathbb{Q}(-1)$. Set $\mathcal{H} := h^1(E)(1)$. It is a simple object of weight -1. The *elliptic motives* are the direct summands of the motives $\mathcal{H}^{\otimes n}$. They form a rigid abelian tensor category $\mathcal{P}_E$.

Example. If E has no complex multiplication then $\mathcal{P}_E$ is equivalent to the category of finite dimensional GL_2-modules over $\mathbb{Q}$. The objects $S^n\mathcal{H}(m)$ are simple and mutually non isomorphic. Any simple object in $\mathcal{P}_E$ is isomorphic to one of them.

The category $\mathcal{M}_E$ of *mixed elliptic motives* is the smallest abelian tensor subcategory of $\mathcal{MM}_k$ which contains $\mathcal{H}$ and is closed under extensions.

For an integer a let $L^*(Sym^n E, a)$ be the leading coefficient of the Taylor expansion of the L-function at $s=a$. For a given pair of integers n, m there are the following intimately related problems:

Problem A. *Find explicit formulas for special values of the L-functions of pure elliptic motives, i.e.*

$$L^*(Sym^n E, n+m) \qquad E/k, \quad k \quad \textit{is a number field}$$

Problem B. *Construct explicitly elliptic motivic complexes*

$$RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m)), \quad k \quad \textit{is an arbitrary field} \qquad (1)$$

They may be non zero only if $-n-2m$, the weight of the motive $Sym^n\mathcal{H}(m)$, is negative.

Problem C. *Find a precise description of the Galois group of the category of mixed elliptic motives.*

A weaker version of the Problem A concerns the special values of L-functions up to a $\mathbb{Q}^*$-factor. We will refer to it as Problem A*.

We are making sense of the motivic Ext's in a usual way. Namely, let $E^{(n)}$ be the kernel of the sum map

$$E^{n+1} \longrightarrow E, \qquad (x_1,...,x_{n+1}) \longmapsto x_1+...+x_{n+1} \tag{2}$$

For a group A living on E^{n+1} the notation A_{sgn} means the part alternating under the action of the group S_{n+1}. Beilinson's description of $Ext^i_{\mathcal{MM}_X}(\mathbb{Q}(0),\mathbb{Q}(n))$ implies (see lemma 3.4) that one should have

$$Ext^i_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m)) = gr^{\gamma}_{n+m}K_{n+2m-i}(E^{(n)})_{sgn}\otimes\mathbb{Q} \tag{3}$$

(Here γ is the γ-filtration on the K-groups). If k is a number field these groups are expected to be zero for $i>1$. Beilinson's regulator map

$$Ext^1_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m)) \longrightarrow Ext^1_{\mathbb{R}-\mathcal{MHS}}(\mathbb{R}(0), S^n\mathcal{H}(m))$$

is provided by the realization functor from $\mathcal{MM}_k$ to the category $\mathbb{R}-\mathcal{MHS}$ of mixed Hodge structures over $\mathbb{R}$.

Problem A is the deepest one. It is of arithmetic nature, while Problems B and C are geometric. We expect such a solution of Problem C that gives the desired answer for Problem B. This answer in the case of number fields should resolve Problem A*. The regulator map should be constructed first over $\mathbb{C}$, brining analysis into the picture and providing a key for Problem B.

In these problems one can consider the category of mixed motives generated by powers of any simple pure motive X. However the case of mixed elliptic motives seems to be especially interesting.

To smell the flavor of the problems let us look at similar questions for the category of mixed Tate motives over a field k (i.e. the rigid tensor subcategory of $\mathcal{MM}_k$ generated by the Tate motive $\mathbb{Q}(1)$).

Then Problem A is about special values $\zeta_k(n)$ of the Dedekind zeta function of a number field k. The ideal answer to Problem A* is given by Zagier's conjecture [Z2].

In Problem B we are looking for complexes $RHom_{\mathcal{MM}_k}(\mathbb{Q}(0),\mathbb{Q}(n))$ for an arbitrary field k. In [G1],[G2] we have constructed complexes $\mathcal{B}(\mathbb{Q}(n);k)^{\bullet}$:

$$\mathcal{B}_n(k) \longrightarrow \mathcal{B}_{n-1}(k)\otimes k^* \longrightarrow ... \longrightarrow \mathcal{B}_2(k)\otimes\Lambda^{n-2}k^* \longrightarrow \Lambda^n k^* \tag{4}$$

which reflect the properties of *classical* n-logarithm function $Li_n(z)$. Here $\mathcal{B}_n(k)$ is the quotient of $\mathbb{Z}[k^*]$ along a certain subgroup $\mathcal{R}_n$, which in the case $k=\mathbb{C}$

is the subgroup of all functional equations for the classical n-logarithm function. It is placed in degree 1. These complexes $\otimes_{\mathbb{Q}}$ were conjectured to be quasiisomorphic to $RHom_{\mathcal{MM}_k}(\mathbb{Q}, \mathbb{Q}(n))$.

A detailed exposition of this philosophy in the two simplest cases, for the motives $\mathbb{Q}(2)$ and $\mathcal{H}(1)$, is given in chapter 2.

According to the Tannakian formalism the category of mixed Tate motives a field k is equivalent to the category of finite dimensional representations of its Galois group. The Galois group is a semidirect product of $\mathbb{G}_m$ and a prounipotent algebraic group scheme $U(k)$ over $\mathbb{Q}$. Let $L(k)$ be the Lie algebra of $U(k)$. The action of $\mathbb{G}_m$ provides a grading $L(k)_\bullet = \oplus_{n\geq 1} L(k)_{-n}$ (it is negatively graded thanks to a weight argument).

The answer to the Problem C for mixed Tate motives which we have in mind is this. Let $I(k)_\bullet := \oplus_{n\geq 2} L(k)_{-n}$. It is an ideal, and $L(k)_\bullet / I(k)_\bullet = (k^*_{\mathbb{Q}})^\vee$. (Here $V \to V^\vee$ is a duality between the inductive and projective limits of finite dimensional $\mathbb{Q}$-vector space.)

The Freeness Conjecture. ([G1],[G2]) *$I(k)_\bullet$ is a free graded pro-Lie algebra generated by the groups $\mathcal{B}_n(k)^\vee$, $n \geq 2$, sitting in degree n.*

For a more precise version see Conjecture 1.20 in [G2], where we proved that it is equivalent to the description (4) of the motivic complexes.

There are several other candidates for the motivic complexes, for example Bloch's beautiful complexes of higher Chow groups [Bl] and their versions. However the complexes (4) are the smallest possible and the only ones directly related just to the classical polylogarithms. For a "cycle" construction of the motivic Lie algebra see [BK].

In this paper I will formulate an elliptic analog of the freeness conjecture, see conjecture 1.2 below.

The Tannakian formalism tells us that there exists a pro-Lie algebra $L(E)$ in the tensor category $\mathcal{P}_E$ such that the category of mixed elliptic motives is equivlent to the category of modules over $L(E)$ in the category $\mathcal{P}_E$. There is a different tensor structure $\otimes'$ on the category of pure elliptic motives. Assume that E has no complex multiplication. Then $S^n\mathcal{H}(m) \otimes' S^{n'}\mathcal{H}(m') = S^{n+n'}\mathcal{H}(m+m')$. Denote by $\mathcal{P}^*_E$ the category of pure elliptic motives with *this* tensor structure.

Theorem 1.1 *Let us assume for simplicity that E is not a CM curve. Then there exists a differential graded pro-Lie algebra $\tilde{L}^*(E)$ in the category $\mathcal{P}^*_E$ such that for $(m,n) \neq (0,0)$ one has*

$$H^i_{\mathcal{P}^*_E}(\tilde{L}^*(E))_{S^n\mathcal{H}(m)^\vee} = gr^\gamma_{n+m} K_{n+2m-i}(E^{(n)})_{sgn} \otimes \mathbb{Q}$$

Here V_M is the M-isotypical component of a simple object V in $\mathcal{P}^*_E$. The DG pro-Lie algebra $\tilde{L}^*(E)$ is constructed in s.4.4.

Conjecture 1.2 *Assume that $k = \bar{k}$.*

a) $L(E)$ has a unique quotient $L^*(E)$ which is a pro-Lie algebra with respect to the both tensor structures on the category of pure elliptic motives and $H^{\bullet}_{\mathcal{P}^*_E}(L^*(E)) = H^{\bullet}_{\mathcal{P}_E}(L(E))$.

b) There is an ideal $I^*(E) \subset L^*(E)$ with an abelian quotient

$$L^*(E)/I^*(E) = (k^*)^{\vee} \boxtimes \mathcal{H} \oplus J^{\vee} \boxtimes \mathcal{H}$$

which should be a free Lie algebra in the tensor category $\mathcal{P}^*_E$.

More about the Lie algebra $L^*(E)$ in s. 4.8. The conjectures about $L^*(E)$ imply a very specific structure for the elliptic motivic complexes.

Assuming the following vanishing conjecture (a version of the Beilinson-Soule conjecture; the left hand side is defined via (3))

$$Ext^i_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m)) = 0 \quad \text{for } i < 0$$

one can show that $H^j_{\partial}(\tilde{L}^*(E)) = 0$ for $j < 0$. Here ∂ is the differential in $\tilde{L}^*(E)$.

Conjecture 1.3 *Assume $k = \bar{k}$.*

a) $H^j_{\partial}(\tilde{L}^*(E)) = 0$ for $j \neq 0$.

b) $H^0_{\partial}(\tilde{L}^*(E))$ is isomorphic to the pro-Lie algebra $L^*(E)$ anticipated in conjecture 1.2.

Remark. Let G be a reductive group. Then there is a new tensor structure the category of finite dimensional G-modules given by $V_\lambda \otimes V_\mu \stackrel{=}{\longrightarrow} V_{\lambda+\mu}$, where V_λ is the G-module with the highest weight λ. Notice that the category of all pure motives is equivalent to the category of finite dimensional modules over a pro-reductive group.

2. The complexes $B(E, n)^{\bullet}$. In chapters 5-7 we study Problems A^* and B in the case $m = 1$. We construct a complex $B(E, n+2)^{\bullet}$ which is conjecturaly quasiisomorphic to $RHom_{\mathcal{MM}_k}(\mathbb{Q}, Sym^n\mathcal{H}(1))$. When E is an elliptic curve over a number field k this leads to a precise conjecture on special values $L(Sym^n E, n+1)$.

This complex is an *elliptic deformation* of the complex (4): if E is a nodal curve and $k = \bar{k}$, it is quasiisomorphic to (4).

Remark. A better notation for the group $B_{n+2}(E)$ and the complex $B(E, n+2)^{\bullet}$ would be $B_{S^n\mathcal{H}(1)}$ and $B(S^n\mathcal{H}(1))^{\bullet}$.

Let us assume first that $k = \bar{k}$. Let $\mathbb{Z}[X]$ be the free abelian group generated by a set X, $\{x\}$ are the generators. We will define subgroups

$$R_n(E/k) \subset \mathbb{Z}[E(k)], \qquad \text{and set} \quad B_n(E) := \frac{\mathbb{Z}[E(k)]}{R_n(E/k)}$$

Put formally $B_0(E) = \mathbb{Z}$. By definition $R_1(E)$ is generated by the elements $\{x\} + \{y\} - \{x + y\}$ where $x, y \in E(k)$. So $B_1(E) = J(k)$, where J is the Jacobian of E.

Define a homomorphism $\delta : \mathbb{Z}[E(k)] \longrightarrow \mathbb{Z}[E(k)] \otimes J(k)$ by the formula $\{a\} \longmapsto \{a\} \otimes a$. We prove in chapter 5 that $\delta(R_n(E)) \subset R_{n-1}(E) \otimes J$, so we get a homomorphism $\delta : B_n(E) \longrightarrow B_{n-1}(E) \otimes J$. Consider the following cohomological complex

$$B_n(E) \longrightarrow B_{n-1}(E) \otimes J \longrightarrow ... \longrightarrow B_1(E) \otimes \Lambda^{n-1} J \longrightarrow B_0(E) \otimes \Lambda^n J \quad (5)$$

where the differential is given by the formula

$$\delta : \{a\} \otimes b_1 \wedge b_2 \wedge ... \wedge b_m \longmapsto \{a\} \otimes a \wedge b_1 \wedge b_2 \wedge ... \wedge b_m$$

It is the complex $B(E, n+1)^\bullet$. We put it in degrees $[1, n+1]$. The complex is acyclic in the last two terms. To emphasize dependency on the ground field k we use a notation $B(E/k, n+1)^\bullet$. If k is not an algebraically closed field we postulate the Galois descent property: $B(E/k, n)^\bullet := \Big(B(E/\bar{k}, n)^\bullet\Big)^{Gal(\bar{k}/k)}$. The complexes $B(E/k, n)^\bullet$ for $n = 2, 3$ were constructed in [GL].

Conjecture 1.4 *a) There exists a canonical homomorphism*

$$H^i\Big(B(E/k, n+2)^\bullet_{\mathbb{Q}}\Big) \longrightarrow gr^\gamma_{n+1} K_{n+2-i}(E^{(n)})_{sgn} \otimes \mathbb{Q} \quad (6)$$

b) This homomorphism is an isomorphism.

Here "canonical" means compatibility with the regulator map, see below. The most nontrivial is the "surjectivity conjecture" from part b).

In [W] J.Wildeshaus, assuming standard conjectures on mixed motives, gave a conjectural inductive definition of groups similar to $H^1 B(E, n+2)^\bullet$ and formulated a conjecture similar to the part a) of conjecture (1.4) for $i = 1$. See also some new developments in this direction in the recent preprint [W2].

In chapter 7 we construct, assuming the standard conjectures, groups $\mathcal{B}_n(E)$. One has a surjective map $B_n(E) \longrightarrow \mathcal{B}_n(E)$ which is hope to be an isomorphism. These groups form a complex $\mathcal{B}(E; n)^\bullet$ similar to the complex (5). We show that the motivic formalism implies that (for $k = \bar{k}$) this complex maps to the standard cochain complex of the motivic Lie algebra $L(E)$, providing a morphism of complexes (for an arbitrary field k)

$$B(E/k, n+2)^\bullet_{\mathbb{Q}} \longrightarrow RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n \mathcal{H}(1)) \quad (7)$$

Conjecture 1.5 *The morphism (7) is a quasiisomorpism.*

This conjecture implies conjecture (1.4).

3. The Eisenstein-Kronecker series, groups $B_{n+1}(E)$ and regulators. The group $R_{n+1}(E/\mathbb{C})$ is a subgroup of the functional equations for the elliptic n-logarithm studied by Bloch for $n = 2$ [Bl] and by Beilinson and Levin [BL] in general. Its real version is the Eisenstein - Kronecker series known from the XIX-th century [We]. These are the special functions we need in our conjecture on $L(Sym^n E, n+1)$. Let us say a few words about them.

Let E be an elliptic curve over $\mathbb{C}$. Choose a holomorphic differential ω on $E(\mathbb{C})$. Let $\Gamma \in \mathbb{C}$ be the lattice of periods of ω. We will always normalize ω in such a way that $\Gamma := \mathbb{Z} \oplus \mathbb{Z}\tau$ where $Im\tau > 0$. The differential ω defines via the Abel-Jacobi map an isomorphism $E(\mathbb{C}) \longrightarrow \mathbb{C}/\Gamma$. The intersection form on $\Gamma = H_1(E(\mathbb{C}), \mathbb{Z})$ provides a pairing

$$(\cdot,\cdot) : E(\mathbb{C}) \times \Gamma \longrightarrow S^1; \qquad (z,\gamma) := exp(\frac{2\pi i(z\bar{\gamma} - \bar{z}\gamma)}{\tau - \bar{\tau}}) \tag{8}$$

Consider the Eisenstein-Kronecker series

$$K_{i,j}(z;\tau) = \sum_{\gamma \in \Gamma \backslash 0} \frac{(z,\gamma)}{\gamma^i \bar{\gamma}^j}, \quad i,j \geq 1$$

There is a homomorphism $K_n : \mathbb{Z}[E(\mathbb{C})] \longrightarrow Sym^{n-2} H^1_B(E(\mathbb{C}), \mathbb{C})$ given by the formulas

$$\{z\} \longmapsto \sum_{a+b=n} K_{a,b}(z) \cdot (dz)^{a-1} (d\bar{z})^{b-1}$$

Theorem (6.2) below claims that it sends the subgroup $R_n(E(\mathbb{C}))$ to zero.

Suppose E is defined over $\mathbb{R}$. Choose a differential $\omega \in \Omega^1(E_{/\mathbb{R}})$ over $\mathbb{R}$. Then the lattice of periods Γ is invariant under the action of complex conjugation $z \longmapsto \bar{z}$ and the isomorphism $E(\mathbb{C}) \longrightarrow \mathbb{C}/\Gamma$ is compatible with complex conjugation.

For any lattice Γ one has $\bar{K}_{i,j}(z;\tau) = (-1)^{i+j} K_{j,i}(z;\tau)$. If a lattice Γ and a divisor P on $\mathbb{C}/\Gamma$ are invariant under complex conjugation, then $K_{i,j}(P;\tau) \in \mathbb{R}$. In this situation (+ means invariants under the complex conjugation acting on $E(\mathbb{C})$ and $\mathbb{R}(1)$) we find a map

$$K_n : B_n[E(\mathbb{C})]^+ \longrightarrow Sym^{n-2} H^1(E(\mathbb{C}), \mathbb{R}(1))^+$$

Restricting K_n to $Ker(B_n(E/\mathbb{C}) \to B_{n-1}(E/\mathbb{C}) \otimes J(\mathbb{C}))$ and assuming conjecture (1.4) we should get the regulator map

$$gr^{\gamma}_{n-1} K_{n-1}(E^{(n-2)})_{sgn} \otimes \mathbb{Q} \longrightarrow Sym^{n-2} H^1(E(\mathbb{C}), \mathbb{R}(1))^+$$

This together with Beilinson's conjecture on regulators implies a precise conjecture on $L(Sym^n E, n+1)$ for elliptic curves over number fields, see chapter 6.

4. The structure of the paper. In chapter 2 we recall explicit construction of the Bloch-Suslin complex ([S]) which essentially computes $RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), \mathbb{Q}(2))$ and its elliptic analog ([GL]) which does the job for $RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), \mathcal{H}(1))$. These constructions and results in the special case when k is a number field lead to explicit formulas for the special values of the Dedekind ζ-function and the L-function of elliptic curves at $s = 2$. In chapter 3 we spell out the basic formalism of motivic Galois groups and motivic Lie algebras. These chapters are expository.

In chapter 4 we prove theorem 1.1 and formaluate some conjectures on the structure of the motivic Lie algebra of the category of mixed elliptic motives. Let me mention three of them: conjecture 4.2 on the Ext groups in the category of mixed elliptic motives, conjecture 4.3 about the small motivic Lie algebra $L^*(E)$ and the freeness conjecture 4.10 for $L^*(E)$.

In chapter 5 we construct the complexes $B(E, n+2)^\bullet$ and $B^*(E, n+2)^\bullet$ (they are canonically quasiisomorphic). In chapter 6 a conjecture on $L(Sym^n E, n+1)$ is formulated. In chapter 7 we show that restriction of the Eisenstein-Kronecker series K_{n+1} to the $2n$-th power of the augmentation ideal in $\mathbb{Z}[E]$ is a real period of an explicitely constructed mixed elliptic motive. This motivic interpretaion seems to be quite different (and simpler) then the one suggested in [BL]. We show how this plus the motivic formalism of chapter 3 allow us to deduce conjecture 1.4a) from standard conjectures on mixed motives. In chapter 8 we define the single valued and multivalued elliptic Chow polylogarithms and introduce the generalized Eisenstein-Kronecker series. We hope to develope further to the material of this chapter in future.

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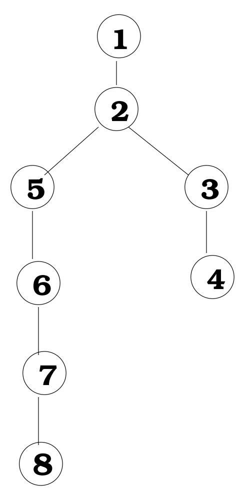


2 Two basic examples: a survey

1. The $\mathbb{Q}(1)$-story: the Bloch-Suslin complex and $\zeta_F(2)$. i) Let $R_{\mathbb{Q}(2)}(k)$ is the subgroup of $\mathbb{Z}[k^*]$ generated by the elements $\sum_i (-1)^i\{r(x_1,...,\hat{x}_i,...,x_5)\}$, where x_i runs through all 5-tuples of distinct points over k on the projective line, and r is the cross ratio. The Bloch-Suslin complex $B(\mathbb{Q}(2);k)^\bullet$ for an arbitrary field k is defined as follows:

$$B_{\mathbb{Q}(2)}(k) \stackrel{\delta}{\longrightarrow} \Lambda^2 k^*; \qquad B_{\mathbb{Q}(2)}(k) := \frac{\mathbb{Z}[k^*]}{R_{\mathbb{Q}(2)}(k)}; \quad \delta : \{x\} \longmapsto (1-x)\wedge x$$

We put $B_{\mathbb{Q}(2)}(k)$ in degree 1. Then one should have a canonical isomorphism in the derived category

$$B(\mathbb{Q}(2);k)^\bullet \quad \stackrel{qis}{=} \quad RHom_{\mathcal{MM}_k}(\mathbb{Q}(0),\mathbb{Q}(2))$$

The main reason for this is the following results describing the relation of the cohomology of the complex $B(\mathbb{Q}(2);k)^\bullet \otimes \mathbb{Q}$ with weight 2 motivic cohomology.

According to Matsumoto's theorem $H^2 B(\mathbb{Q}(2);k)^\bullet \quad = \quad K_2(k)$.

Suslin proved ([S], see also [DS]) that

$$H^1 B(\mathbb{Q}(2);k)^\bullet \otimes \mathbb{Q} \quad = \quad K_3^{ind}(k)\otimes \mathbb{Q} \quad = \quad gr_2^\gamma K_3(k)\otimes \mathbb{Q}$$

In fact, Suslin proved in [S] that the following sequence is exact:

$$0 \longrightarrow \tilde{T}or(k^*,k^*) \longrightarrow K_3^{ind}(k) \longrightarrow B_{\mathbb{Q}(2)}(k) \stackrel{\delta}{\longrightarrow} \Lambda^2 k^* \longrightarrow K_2(k) \longrightarrow 0$$

Here $\tilde{T}or(k^*,k^*)$ is a nontrivial extension of $Tor(k^*,k^*)$ by $\mathbb{Z}/2\mathbb{Z}$.

Finally, there exists a canonical morphism in the derived category

$$B(\mathbb{Q}(2);k)^\bullet \otimes \mathbb{Q} \longrightarrow \tau_{\geq 1} RHom_{\mathcal{MM}_k}(\mathbb{Q}(0),\mathbb{Q}(2))$$

inducing the isomorphism on H^1 and H^2. The right hand side can be understood, for example, as the complex of Bloch's higher Chow groups. The existence of such a morphism follows also from the motivic philosophy and the motivic construction of the dilogarithm in [BGSV]. According to the Beilinson-Soulé vanishing conjecture one should have

$$gr_2^\gamma K_{4+i}(k)\otimes \mathbb{Q} = 0 \quad \text{for } i \geq 0$$

One can reformulate this by saying that the canonical morphism

$$RHom_{\mathcal{MM}_k}(\mathbb{Q}(0),\mathbb{Q}(2)) \longrightarrow \tau_{\geq 1} RHom_{\mathcal{MM}_k}(\mathbb{Q}(0),\mathbb{Q}(2))$$

is a quasiisomorphism

The Bloch-Suslin complex is definitely "the smallest possible" representative of the motivic complex $RHom_{\mathcal{MM}_k}(\mathbb{Q}(0),\mathbb{Q}(2))$. I think it is also "the best possible", but this is of cource matter of taste. It gives a solution of the Problem B for the motive $\mathbb{Q}(2)$ which we have in mind.

ii) Now let $k = \mathbb{C}$. Recall that the dilogarithm is the following multivalued analytic function of on $\mathbb{C}P^1 \backslash \{0, 1, \infty\}$):

$$Li_2(z) = -\int_0^z \log(1-t)\frac{dt}{t}$$

It has a single-valued version, the Bloch-Wigner function:

$$\mathcal{L}_2(z) := ImLi_2(z) + \arg(1-z) \cdot \log|z|$$

Let

$$\tilde{\mathcal{L}_2} : \mathbb{Z}[\mathbb{C}^*] \longrightarrow \mathbb{R}, \quad \{z\} \longmapsto \mathcal{L}_2(z)$$

Then one can show that $\tilde{\mathcal{L}}_2(R_{\mathbb{Q}(2)}(\mathbb{C})) = 0$, so we get a well defined homomorphism

$$\tilde{\mathcal{L}_2} : B_{\mathbb{Q}(2)}(\mathbb{C}) \longrightarrow \mathbb{R}$$

Theorem 2.1 *The restriction of the homomorphism $\tilde{\mathcal{L}}_2$ to the subspace $H^1 B(\mathbb{Q}(2); k)^\bullet \otimes \mathbb{Q} = K_3^{ind}(\mathbb{C}) \otimes \mathbb{Q}$ coinsides with the Borel regulator on $K_3^{ind}(\mathbb{C}) \otimes \mathbb{Q}$*

A simple direct proof of this result see in [G1].

Finally, if $k = F$ is a number field combining the two theorems above with the Borel theorem we get the solution of the Problem A* for $\zeta_F(2)$:

Theorem 2.2 *a) There exists elements $y_1, ..., y_r \in H^1 B(\mathbb{Q}(2); F)^\bullet \otimes \mathbb{Q}$ such that*

$$q \cdot \zeta_F(2) = \pi^{2(r_1+r_2)} d_F^{-1/2} det\Big(\tilde{\mathcal{L}}_2(\sigma_i(y_j))\Big)$$

for certain $q \in \mathbb{Q}^*$.

b) For any $y_1, ..., y_r \in H^1(B(\mathbb{Q}(2); F)^\bullet) \otimes \mathbb{Q}$ one has the formula above with $q \in \mathbb{Q}$.

To solve the Problem A one should be able to determine *effectively* the constant $q(y_1 \wedge ... \wedge y_r)$. I do not know how to do this. So we see that the analysis (the dilogarithm) indeed suggested a key for the solution of the Problem B, which leads to a solution of the Problem A*. The relation with the Problem C was explained in details in [G2] and will be scketched briefly in section 4.

Remark. The standard notation for the groups $B_{\mathbb{Q}(n)}(k)$ is $B_n(k)$.

2. The $\mathcal{H}(1)$-story: the elliptic deformation of the Bloch-Suslin complex and $L(E, 2)$ ([GL]) . Let us suppose that k is an algebraically closed field. Let I_E be the augmentaion ideal of the group algebra $\mathbb{Z}[E(k)]$, and I_E^4 its fourth pour. Let B_3^* be the quotient of I_E^4 by the subgroup generated by the elements $(f) * (1-f)^-$, where $*$ is the convolution in the group algebra $\mathbb{Z}[E]$, $f \in k(E)^*$, and $g^-(t) := g(-t)$. Then there is a map

$$\delta^*_{\mathcal{H}(1)} : B_3^*(E) \longrightarrow k^* \otimes J \tag{9}$$

whose construction will be outlined below. The complex we get putting the left group in the degree 1 is called the complex $B^*(E,3)^\bullet$. This is our candidate for $RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), \mathcal{H}(1))$.

If $k = \mathbb{C}$ we can "define" $\delta^*_{\mathcal{H}(1)}$ as follows:

$$\sum n_i\{a_i\} \longmapsto \sum n_i \theta(a_i) \otimes a_i$$

where

$$\theta(z) = \prod_{n\in\mathbb{Z}} (1-q^n z) := q^{1/12} z^{-\frac{1}{2}} \prod_{j\geq 0}(1-q^j z)\prod_{j>0}(1-q^j z^{-1})$$

To make sense out of this formula for arbitrary field k we proceed as follows (see also s. 4.6 of [GL]).

i. The group $B_2(E)$. Let E be an elliptic curve over an arbitrary field k. In s.2.1-2.2 of [GL] we defined a group $B_2(E/k) = B_2(E)$ which fits in the following diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ 0 & \longrightarrow & \mathbb{G}_m(k) & \longrightarrow & B_2(E/k) & \xrightarrow{p} & S^2J(k) & \longrightarrow & 0 \\ & & & & \uparrow\theta & & & & \\ & & & & \mathbb{Z}[E(k)\backslash 0] & & & & \end{array}$$

such that $p\circ\theta : \{a\} \longmapsto a\cdot a$. here the horisontal sequence is exact, and the vertical is exact at least if $k = \bar{k}$. The map θ is defined modulo 6-torsion.

Moreover, if K is a local field there exists a canonical homomorphism

$$h : B_2(E/K) \longrightarrow \mathbb{R}$$

whose restriction to the subgroup $K^* \subset B_2(E/K))$ is given by $x \longmapsto \log|x|$, (see s. 2.3). The canonical local height h_K is given by the composition

$$\mathbb{Z}[E(K)\backslash 0] \xrightarrow{\theta} B_2(E/K) \xrightarrow{h} \mathbb{R}$$

The group $B_2(E)$ appears naturally from the theory of biextensions. Namely, let $\mathcal{P}$ be the rigidified Poincare line bundle over $J\times J$: its fiber over $(0,0)$ is identified with k^*. The restriction of $\mathcal{P}$ to $x\times J$ (resp. $J\times y$) minus the zero section has canonical structure of a commutative algebraic group over k. It is the extension of J by $\mathbb{G}_m$ corresponding to x (resp y). This means that for every point $(x,y) \in J \times J$ we have a k^* torsor $T_{(x,y)}$ together with morphisms of k^*-torsors

$$T_{(x_1,y)} \otimes_{k^*} T_{(x_2,y)} \longrightarrow T_{(x_1+x_2,y)} \qquad T_{(x,y_1)} \otimes_{k^*} T_{(x,y_2)} \longrightarrow T_{(x,y_1+y_2)}$$

providing the group structure "in horisontal and vertical directions" such that the diagram

$$\begin{array}{ccc} T_{(x_1,y_1)} \times T_{(x_2,y_1)} \times T_{(x_1,y_2)} \times T_{(x_2,y_2)} & \longrightarrow & T_{(x_1,y_1+y_2)} \times T_{(x_2,y_1+y_2)} \\ \downarrow & & \downarrow \\ T_{(x_1+x_2,y_1)} \times T_{(x_1+x_2,y_2)} & \longrightarrow & T_{(x_1+x_2,y_1+y_2)} \end{array}$$

is commutative. The horisontal sequence in the diagram above is just a different way to spell these properties of the torsors $T_{(x,y)}$. Set $\{a\}_2 := \theta(\{a\}) \in B_2(E)$.

ii) The group $B_3^*(E)$. Consider the homomorphism $(J := J(k))$

$$\delta_{\mathcal{H}(1)} : \mathbb{Z}[E(k)] \longrightarrow B_2(E) \otimes J, \quad \{a\} \longrightarrow -\frac{1}{2}\{a\}_2 \otimes a$$

Let $R_3 \subset \mathbb{Z}[E(k)]$ be the subgroup generated by R_3^* and the distribution relations

$$m \cdot (\{a\} - m \cdot \sum_{mb=a} \{b\}), \quad a \in E(k), \quad m = -1, 2$$

Then $\delta_{\mathcal{H}(1)}(R_3(E)) = 0$ by theorem (3.3) in [GL]. Setting

$$B_3(E) := \frac{\mathbb{Z}[E(k)]}{R_3(E)}$$

we get a homomorphism $\delta_{\mathcal{H}(1)} : B_3(E) \longrightarrow B_2(E) \otimes J$. There is a map $i : B_3^*(E) \longrightarrow B_3(E)$ provided by the inclusion $i : I_E^4 \hookrightarrow \mathbb{Z}[E]$. We define $\delta_{\mathcal{H}(1)}$ as a morphism making commutative the diagram

$$\begin{array}{ccc} B_3^*(E) & \stackrel{\delta^*_{\mathcal{H}(1)}}{\longrightarrow} & k^* \otimes J \\ \downarrow & & \downarrow \\ B_3(E) & \stackrel{\delta_{\mathcal{H}(1)}}{\longrightarrow} & B_2(E) \otimes J \end{array}$$

Let us consider the following complex

$$B(E;3)^\bullet : \qquad B_3(E) \stackrel{\delta_{\mathcal{H}(1)}}{\longrightarrow} B_2(E) \otimes J \longrightarrow J \otimes \Lambda^2 J \longrightarrow \Lambda^3 J \qquad (10)$$

Here the middle arrow is $\{a\}_2 \otimes b \longmapsto a \otimes a \wedge b$ and the last one is the canonical projection. The complex is placed in degrees $[1,4]$. It is acyclic in the last two terms. It was proved in [GL] that we get the commutative diagram:

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
B_3^*(E) & \stackrel{\delta^*_{\mathcal{H}(1)}}{\longrightarrow} & k^*\otimes J & & & & \\
\downarrow & & \downarrow & & & & \\
B_3(E) & \stackrel{\delta_{\mathcal{H}(1)}}{\longrightarrow} & B_2(E)\otimes J & \longrightarrow & J\otimes\Lambda^2 J & \longrightarrow & \Lambda^3 J \\
\downarrow & & \downarrow & & \downarrow = & & \downarrow = \\
S^3 J & \longrightarrow & S^2 J\otimes J & \longrightarrow & J\otimes\Lambda^2 J & \longrightarrow & \Lambda^3 J \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

where the vertical sequences are exact, and the bottom one is the Koszul complex, and thus also exact. So $B^*(E;3)^\bullet$ is canonically quasiisomorphic to $B(E;3)^\bullet$. If $k\neq\bar{k}$ we postulate the Galois descent:

$$B^*(E/k;3)^\bullet_{\mathbb{Q}} := B^*(E/\bar{k};3)^{\bullet\,Gal(\bar{k}/k)}_{\mathbb{Q}}$$

iii) The elliptic dilogarithm, algebraic K-theory and regulator on $K_2(E/\mathbb{C})$. Let $E(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$ be the complex points of an elliptic curve E. Here $q := exp(2\pi i\tau), Im\tau > 0$.

$$\mathcal{L}_{2,q}(z) := \sum_{n\in\mathbb{Z}} \mathcal{L}_2(q^n z), \qquad \mathcal{L}_{2,q}(z^{-1}) = -\mathcal{L}_{2,q}(z)$$

(The series converges since $\mathcal{L}_2(z)$ has a singularity of type $|z|\log|z|$ near $z=0$ and $\mathcal{L}_2(z) = -\mathcal{L}_2(z^{-1})$). When $k=\mathbb{C}$ the group $R_3(E)$ is the subgroup of all functional equations for the elliptic dilogarithm. In particular the homomorphism

$$\tilde{\mathcal{L}}_{2,q} : \mathbb{Z}[E(\mathbb{C})] \longrightarrow \mathbb{R}, \quad \{a\} \longmapsto \mathcal{L}_{2,q}(a)$$

annihilates the subgroup $R_3(E/\mathbb{C})$.

Let $K(E)^-$ be the minus part of the action of the involution $x\to -x$, $x\in E$.

Theorem 2.3 **[GL]** *a) Let $k=\bar{k}$. Then there is a sequence*

$$0\longrightarrow Tor(k^*,J)\longrightarrow gr_2^\gamma K_2(E)^- \longrightarrow B_3^*(E)\longrightarrow k^*\otimes J\longrightarrow gr_2^\gamma K_1(E)^- \to 0$$

It is exact modulo 2-torsion (and exact in $k^*\otimes J$).

b) Now let $k=\mathbb{C}$. Then the composition

$$gr_2^\gamma K_2(E)^- \longrightarrow B_3^*(E) \stackrel{\tilde{\mathcal{L}}_{2,q}}{\longrightarrow} \mathbb{R}$$

coinsides with the Bloch - Beilinson regulator map.

Since we have the Galois descent on $gr_2^\gamma K_i(E)^-_{\mathbb{Q}}$, we immediately get

Corollary 2.4 *Let E be a curve over an arbitrary field k. Then*

$$H^1(B^*(E;3)^\bullet) = gr_2^\gamma K_2(E)_{\mathbb{Q}}^-, \qquad H^2(B^*(E;3)^\bullet) = gr_2^\gamma K_1(E)_{\mathbb{Q}}^-$$

Using the complex $B(E,3)$ instead of $B^*(E,3)$ the following lemma makes the conditions on the divisor P effectively computable (see a numerical example in s.1.3 of [GL]) .

Lemma 2.5 *Let E be an elliptic curve over a number field F. Then an F-rational divisor $P = \sum n_j(P_j)$ over $\bar{\mathbb{Q}}$ belongs to $Ker\Big(B_3(E) \longrightarrow B_2(E) \otimes J\Big)$ if and only if it satisfy the following conditions:*

a)
$$\sum n_j P_j \otimes P_j \otimes P_j = 0 \quad in \quad S^3 J(\bar{\mathbb{Q}}) \tag{11}$$

b) For any valuation v of the field $F(P)$ generated by the coordinates of the points P_j

$$\sum n_j h_v(P_j) \cdot P_j = 0 \quad in \quad J(\bar{\mathbb{Q}}) \otimes \mathbb{R} \tag{12}$$

In particular for any field k the group of k-rational divisors on $E(\bar{\mathbb{Q}})$ satisfying the conditions a)-b) above maps surjectively to $gr_2^\gamma K_2(E_{/k})_{\mathbb{Q}}^-$, and when k is a number field this map is compatible with the regulator map in the obvious way.

Combining lemma 2.5 with Beilinson's conjecture on regulators we come to an explicit formula for $L(E,2)$, which we formulate only for $k = \mathbb{Q}$ leaving the general case to the reader.

Conjecture 2.6 *Let E be an elliptic curve over $\mathbb{Q}$. Then for any $\mathbb{Q}$-rational divisor P on $E(\bar{\mathbb{Q}})$ satisfying the conditions (11) and (12), and an integrality condition at each prime p where E has a split multiplicative reduction (see 47) one has*

$$\pi\tilde{\mathcal{L}}_{2,q}(P) = q \cdot L(E,2) \quad for\ a\ certain\ q \in \mathbb{Q}$$

Adding to the game Beilinson's results on regulators for modular curves we proved this formula for modular elliptic curves over $\mathbb{Q}$ ([GL]):

Theorem 2.7 *Let E be a modular elliptic curve over $\mathbb{Q}$. Then there exists a $\mathbb{Q}$-rational divisor P on $E(\bar{\mathbb{Q}})$ satisfying the conditions of conjecture above such that*

$$\pi\tilde{\mathcal{L}}_{2,q}(P) = q \cdot L(E,2) \quad for\ a\ certain\ q \in \mathbb{Q}^*$$

So for the motive $\mathcal{H}(1)$ we see the same kind of relationship between the Problems A* and B. It remains to see the role of the motivic Galois group of the category of mixed elliptic motives, i.e. to understand

Why the motivic complex $RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), \mathcal{H}(1))$ has the shape (9), how it reflects the structure of the motivic Galois group, and what it tells us about the motivic Galois group?

In particular, how to define the group $B_3^*(E)$ it terms of the motivic Lie algebra of the category $\mathcal{M}_E$? The answers are given in chapter 4 below.

3 Mixed motives and motivic Lie algebras

1. Categories of mixed motives. Let $\mathcal{MM}_X$ be the (hypothetical) abelian category of all mixed motivic sheaves over a regular scheme X. When $X = Spec(k)$, k is an arbitrary field, we get the category $\mathcal{MM}_k$. We will assume that it satisfies all the expected properties conjectured by Beilinson [Be]. In particular any object of $\mathcal{MM}_X$ has a canonical increasing weight filtration $W_\bullet$; morphisms are strickly compatible with $W_\bullet$. We will ignore the fact that existence of such an abelian category is not known yet.

Let $\pi : X \longrightarrow Spec(k)$ be the structure morphism. There are the Tate sheaves $\mathbb{Q}(n)_X := \pi^*\mathbb{Q}(n)_{Spec(k)}$ which we usually denote simply by $\mathbb{Q}(n)$. The basic conjecture is Beilinson's description of Ext's between them:

Conjecture 3.1

$$Ext^i_{\mathcal{MM}(X)}(\mathbb{Q}(0)_X, \mathbb{Q}(n)_X) = gr^\gamma_n K_{2n-i}(X) \otimes \mathbb{Q}$$

Consider the category of pure motives over a field k. One can have in mind Grothendieck's category of motives with morphisms given by the Chow correspondences modulo *numerical* equivalence. We will assume that it is a semisimple abelian category.

Now let $\mathcal{P}$ be a rigid tensor subcategory of the category of pure motives, and $\mathcal{M}_\mathcal{P}$ the tensor category of mixed motives generated by $\mathcal{P}$. This means that $\mathcal{M}_\mathcal{P}$ is closed under extensions, contains $\mathcal{P}$ as a full subcategory and the weight graded quotients of any object of $\mathcal{M}_\mathcal{P}$ belong to $\mathcal{P}$.

Examples. 1. $\mathcal{P}$ is the category of pure Tate motives. $\mathcal{M}_\mathcal{P}$ is the category of mixed Tate motives.

2. $\mathcal{P}$ is the category of pure elliptic motives. Then $\mathcal{M}_\mathcal{P}$ is the category of mixed elliptic motives.

3. $\mathcal{P}$ is the category of all pure motives over k, so $\mathcal{M}_\mathcal{P}$ is the category of all mixed motives.

There is a canonical fiber functor

$$\Psi : \mathcal{M}_\mathcal{P} \longrightarrow \mathcal{P}, \quad M \longmapsto \oplus_{n\in\mathbb{Z}} gr^W_n M$$

Let us axiomatise this situation. Namely, let $\mathcal{P}$ be an abelian semisimple rigid tensor $\mathbb{Q}$-category . We will say that $\mathcal{M}_\mathcal{P}$ is a mixed category over $\mathcal{P}$ if it is an abelian rigid tensor $\mathbb{Q}$-category containing $\mathcal{P}$ as a full subcategory with the following properties:

1. Any object M of $\mathcal{M}_\mathcal{P}$ carries a canonical finite filtration $W_\bullet M$ (the weight filtration).
2. Morphisms are strictly compatible with $W_\bullet$.
3. The graded objects $gr^W_i M$ belong to $\mathcal{P}$.
4. $Hom_{\mathcal{M}_\mathcal{P}}(M, N)$ are finite dimensional.

Remark. We can assume also that $\mathcal{M}_\mathcal{P}$ is an F-category where F is an arbitrary field of characteristic zero. This is essential when E is a curve with complex multiplication by $\mathcal{O}_F$, so $\mathcal{M}_E$ is an F-category. However to simplify a bit notations we will assume $F = \mathbb{Q}$ below.

Below we will assume that $\mathcal{M}_{\mathcal{P}}$ is such a category, not necessarily of motivic origin.

2. The fundamental Lie algebra of a mixed category over $\mathcal{P}$. Set $H(\mathcal{M}_{\mathcal{P}}) := End(\Psi)$. It is a cocommutative Hopf algebra in the tensor category $\mathcal{P}$.

Let $L(\mathcal{M}_{\mathcal{P}})$ be the Lie algebra of all derivations of the functor Ψ:

$$L(\mathcal{M}_{\mathcal{P}}) = Der(\Psi) = \{F \in End(\Psi) | F_{X \otimes Y} = F_X \otimes id_Y + id_X \otimes F_Y\}$$

It is a Lie algebra in the tensor category $\mathcal{P}$. Its universal enveloping algebra is isomorphic to the Hopf algebra $H(\mathcal{M}_{\mathcal{P}})$. The Tannakian formalism shows that the functor Ψ provides an equivalence between the category $\mathcal{M}_{\mathcal{P}}$ and the category of $L(\mathcal{M}_{\mathcal{P}})$-modules in the category $\mathcal{P}$.

A Lie coalgebra in a tensor category $\mathcal{C}$ is an object $\mathcal{L}$ together with a cobracket $\delta : \mathcal{L} \to \Lambda^2_{\mathcal{C}}\mathcal{L}$ such that the composition

$$\mathcal{L} \stackrel{\delta}{\longrightarrow} \Lambda^2_{\mathcal{C}}\mathcal{L} \stackrel{\delta\otimes id - id\otimes\delta}{\longrightarrow} \Lambda^3_{\mathcal{C}}\mathcal{L}$$

is zero. The standard complex of $\mathcal{L}$ is defined as follows:

$$C^\bullet_{\mathcal{C}}(\mathcal{L}) := \mathcal{L} \stackrel{\delta}{\longrightarrow} \Lambda^2_{\mathcal{C}}\mathcal{L} \stackrel{\delta\otimes id - id\otimes\delta}{\longrightarrow} \Lambda^3_{\mathcal{C}}\mathcal{L} \longrightarrow \Lambda^4_{\mathcal{C}}\mathcal{L} \longrightarrow ...$$

Here $\mathcal{L}$ placed in degree 1, and the differential has degree $+1$. We define the cohomology groups of a Lie coalgebra $\mathcal{L}$ setting $H^\bullet_{\mathcal{C}}(\mathcal{L}) := H^\bullet(C^\bullet(\mathcal{L}))$.

There is a duality $V \to V^\vee$, $(V^\vee)^\vee = V$ between the inductive and projective limits of objects in the cateogory $\mathcal{P}$. Set $\mathcal{L}(\mathcal{M}_{\mathcal{P}}) := L(\mathcal{M}_{\mathcal{P}})^\vee$. It is a Lie coalgebra in the tensor category $\mathcal{P}_{\mathcal{M}}$.

For a $\mathbb{Q}$-vector space V and an object X of a category $\mathcal{C}$ there is an object $V \boxtimes X$ of $\mathcal{C}$ such that $Hom_{\mathcal{C}}(Y, V \otimes X) = V \otimes Hom_{\mathcal{C}}(Y, X)$. Any pure motive W can be canonically decomposed into the isotypical components:

$$W = \oplus_M Hom_{\mathcal{M}}(M, W) \boxtimes M$$

where the sum is over the isomorphism classes of simple objects in $\mathcal{M}_{\mathcal{P}}$. The standard complex of $\mathcal{L}(\mathcal{M}_{\mathcal{P}})$ splits into a direct sum of isotypical components

$$\Big(\mathcal{L}(\mathcal{M}_{\mathcal{P}})\Big)_M \stackrel{\delta}{\longrightarrow} \Big(\Lambda^2\mathcal{L}(\mathcal{M}_{\mathcal{P}})\Big)_M \stackrel{\delta}{\longrightarrow} \Big(\Lambda^3\mathcal{L}(\mathcal{M}_{\mathcal{P}})\Big)_M \stackrel{\delta}{\longrightarrow} ... \qquad (13)$$

Notice that $\mathcal{L}(\mathcal{M}_{\mathcal{P}})_M$ is zero unless the weight M is > 0. Therefore each of the complexes (13) has finite length.

3. The Galois group of a mixed category. Let $\Phi : \mathcal{M}_{\mathcal{P}} \longrightarrow Vect_{\mathbb{Q}}$ be composition of the fiber functor Ψ with a fiber functor φ from $\mathcal{P}$ to the category of finite dimensional $\mathbb{Q}$-vector spaces.

Then $G(\mathcal{M}_{\mathcal{P}}) := Aut^{\otimes}\Phi$ is a proalgebraic group scheme over $\mathbb{Q}$. It is the Galois group of the category $\mathcal{M}_{\mathcal{P}}$.

There are two canonical functors between the tensor categories $\mathcal{M}_{\mathcal{P}}$ and $\mathcal{P}$: the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{M}_{\mathcal{P}}$, and the functor

$$gr^{\bullet} : \mathcal{M}_{\mathcal{P}} \longrightarrow \mathcal{P} \qquad gr^{W} : X \longmapsto \oplus_{n\in\mathbb{Z}} gr^{W}_{n} X$$

Their composition is the identity functor on $\mathcal{P}$. These functors obviously respect the fiber functor, and so lead to homomorphisms of groups $G(\mathcal{M}_{\mathcal{P}}) \longrightarrow G(\mathcal{P})$ and $G(\mathcal{P}) \longrightarrow G(\mathcal{M}_{\mathcal{P}})$. Thus the group $G(\mathcal{M}_{\mathcal{P}})$ is a semidirect product:

$$0 \longrightarrow U(\mathcal{M}_{\mathcal{P}}) \longrightarrow G(\mathcal{M}_{\mathcal{P}}) \longrightarrow G(\mathcal{P}) \longrightarrow 0$$

Passing to Lie algebras we get:

$$0 \longrightarrow LieU(\mathcal{M}_{\mathcal{P}}) \longrightarrow LieG(\mathcal{M}_{\mathcal{P}}) \longrightarrow LieG(\mathcal{P}) \longrightarrow 0$$

So $LieU(\mathcal{M}_{\mathcal{P}})$ is a pronilpotent Lie algebra in the category of $G(\mathcal{P})$-modules.

The category of finite dimensional $G(\mathcal{M}_{\mathcal{P}})$-modules is equivalent to the category of $U(\mathcal{M}_{\mathcal{P}})$-modules in the category finite dimensional $G(\mathcal{P})$-modules. Since the group scheme $U(\mathcal{M}_{\mathcal{P}})$ is prounipotent, it is equivalent to the category of $LieU(\mathcal{M}_{\mathcal{P}})$-modules in the category of $G(\mathcal{P})$-modules. One can think about it as of the category $G(\mathcal{P})$-modules equipped with an action of the Lie algebra $LieU(\mathcal{M}_{\mathcal{P}})$ such that the action $LieU(\mathcal{M}_{\mathcal{P}}) \otimes V \longrightarrow V$ is a morphism of $G(\mathcal{P})$-modules.

Lemma 3.2 *Let M be a pure object. Then*

$$RHom_{LieU(\mathcal{M}_{\mathcal{P}})-mod}(\varphi(\mathbb{Q}(0)), \varphi(M)) \boxtimes M^{\vee} = \left(\left(\Lambda^{\bullet}\mathcal{L}(\mathcal{M}_{\mathcal{P}})\right)_{M}, \partial\right)$$

4. A description of the fundamental Lie coalgebra. (Compare with [BMS] and [BGSV]). Let A, B be simple objects of the category $\mathcal{P}$. Let us say that an object M of $\mathcal{M}_{\mathcal{P}}$ is n-framed by A, B if we are given nonzero homomorphisms

$$v_A : A \to gr^{W}_{0} M, \quad f_B : gr^{W}_{-n} M \to B$$

Consider the finest equivalence relation on the set of all objects n-framed by A, B such that $(M; v_A; f_B)$ and $(M'; v'_A; f'_B)$ are equivalent if there is a morphism $M \to M'$ respecting the frames. Denote by $\mathcal{A}(A, B)$ the set of equivalence classes of mixed motives n-framed by A, B.

Let us define an addition law setting

$$(M; v_A; f_B) + (M'; v'_A; f'_B) := (M \oplus M'; (v_A, v'_A); f_B + f'_B)$$

The neutral element is $A \oplus B$ with the obviuos framing. Indeed, the equivalence between $(M \oplus A \oplus B; (v_A, id_A); (f_B + id_B))$ and $(M; v_A; f_B)$ is provided by the natural morphisms $M \longleftarrow M \oplus A \hookrightarrow M \oplus A \oplus B$.

The inversion is given by $-(M; v_A; f_B) := (M; v_A; -f_B)$, and one also has $(M; v_A; -f_B) = (M; -v_A; f_B)$. Let us prove the first claim. The (A, B)-framed object object $W_{\leq 0}M/Ker f_B$ is equivalent to $(M; v_A; f_B)$. So we may assume without loss of generality that $W_{\leq 0}M = M, W_{<-n}M = 0$, and moreover $gr_0^W M = A, gr_{-n}^W M = B$. There is a morphism

$$i_- : W_{<0}M \hookrightarrow M \oplus M, m \longmapsto (m, -m)$$

Then

$$M \oplus M \longrightarrow \frac{M \oplus M}{i_-(M)} =: N_1$$

is a morphism of (A, B)-framed objects. We get an extension

$$0 \longrightarrow W_{<0}N_1 \longrightarrow N_1 \longrightarrow A \oplus A \longrightarrow 0$$

Let us consider the extension $0 \longrightarrow W_{<0}N_1 \longrightarrow N \longrightarrow A \longrightarrow 0$ induced by the morphism $j_- : A \to A \oplus A, a \longmapsto (a, -a)$. Then there is a well defined morphism $\alpha : A \to N$ given by $a \longmapsto (n_a, -n_a)$ where $n_a \in N$ is any element projecting to a. So we get a morphism of (A, B)-framed objects $A \oplus B \stackrel{(\alpha,\beta)}{\longrightarrow} N$, where $\beta : B \hookrightarrow N$ is the natural inclusion.

It is easy to check that $\mathcal{A}(A, B)$ is an abelian group. Since $\mathcal{M}_{\mathcal{P}}$ is a $\mathbb{Q}$-category, it is a $\mathbb{Q}$-vector space, and in fact an inductive limit of finite dimensional F-vector spaces.

If B is a simple object then $\mathcal{A}(\mathbb{Q}(0), B) \boxtimes B^\vee$ is an object of $\mathcal{P}$ defined by the isomorphism class of B up to a *unique* isomorphism. Set

$$H'(\mathcal{M}_{\mathcal{P}}) := \oplus \mathcal{A}(\mathbb{Q}(0), B) \boxtimes B^\vee$$

The sum is over all isomorphism classes of simple objects B in $\mathcal{P}$, but only finitely many terms are non zero.

The tensor product of mixed motives provides $H'(\mathcal{M}_{\mathcal{P}})$ with a structure of a commutative algebra in the tensor category $\mathcal{P}$:

$$\mathcal{A}(\mathbb{Q}(0), B_1) \boxtimes B_1^\vee \otimes \mathcal{A}(\mathbb{Q}(0), B_2) \boxtimes B_2^\vee \to \mathcal{A}(\mathbb{Q}(0), B_1 \otimes B_2) \boxtimes B_1^\vee \otimes B_2^\vee$$

Let us define the coproduct. Let C be a simple pure object of weight $-k$. Let $c \in Hom_{\mathcal{P}}(gr_{-k}^W M, C)$ and $c^* \in Hom_{\mathcal{P}}(C, gr_{-k}^W M)$. Let

$$v(c^*): \quad \mathbb{Q}(0) \longrightarrow C \otimes C^\vee \stackrel{c^* \otimes id}{\longrightarrow} gr_{-k}^W M \otimes C^\vee = gr_0^W (M \otimes C^\vee)$$

(The first arrow is the canonical morphism). Set

$$\alpha(c) := (M; v_{\mathbb{Q}(0)}; c) \in \mathcal{A}(\mathbb{Q}(0), C) \boxtimes C^\vee$$

$$\alpha^*(c^*) := (M \otimes C^\vee; v(c_i^*); f_B \otimes id_{C^\vee}) \in \mathcal{A}(\mathbb{Q}(0), C^\vee \otimes B) \boxtimes C \otimes B^\vee$$

Define a natural map

$$\nu_C : Hom_{\mathcal{P}}(gr^W_{-k}M, C) \otimes Hom_{\mathcal{P}}(C, gr^W_{-k}M) \longrightarrow \qquad (14)$$

$$\Big(\mathcal{A}(\mathbb{Q}(0), C) \boxtimes C^\vee\Big) \otimes \Big(\mathcal{A}(\mathbb{Q}(0), C^\vee \otimes B) \boxtimes C \otimes B^\vee\Big)$$

by setting

$$\nu_C : c \otimes c^* \longmapsto \alpha(c) \otimes \alpha^*(c^*)$$

There is a canonical element in (14) defined as follows. Let $c_1, ..., c_m$ be a basis of the $\mathbb{Q}$-vector space $Hom_{\mathcal{P}}(gr^W_{-k}M, C)$ and $c_1^*, ..., c_m^*$ the dual basis of $Hom_{\mathcal{P}}(C, gr^W_{-k}M)$, i.e. $c_i \circ c_j^* = \delta_{i,j} Id_C$. Then $\sum_i c_i \otimes c_i^*$ does not depend on the choice of basis c_i. By definition

$$\Delta : \quad (M; v_{\mathbb{Q}(0)}; f_B) \boxtimes B^\vee \longmapsto \oplus_C \nu_C(\sum_i c_i \otimes c_i^*) \quad \in$$

$$\oplus_C \Big(\mathcal{A}(\mathbb{Q}(0), C) \boxtimes C^\vee\Big) \otimes \Big(\mathcal{A}(\mathbb{Q}(0), C^\vee \otimes B) \boxtimes C \otimes B^\vee\Big)$$

The sum is over all isomorphism classes of simple objects in $\mathcal{P}$ whose weights are between 0 and $wt(B)$. It turnes out to be a finite sum.

Notice that $V := \oplus V_B \boxtimes B^\vee$ is a pro (resp. ind) object in $\mathcal{P}$ if and only if V_B is pro (resp. ind) $\mathbb{Q}$-vector space for each isomorphism class of simple objects in $\mathcal{P}$. There is a duality between the pro- and ind-objects: if V is a pro (resp. ind) object in $\mathcal{P}$, then $V^\vee := \oplus V_B^\vee \boxtimes B$ is an ind (resp. pro) object in $\mathcal{P}$, and the canonical map $V \to (V^\vee)^\vee$ is an isomorphism.

For any simple object B of $\mathcal{P}$ the B-isotypical componenmt of $H'(\mathcal{M}_{\mathcal{P}})$ is an ind-object in the category $\mathcal{P}$. This means that $H'(\mathcal{M}_{\mathcal{P}})$ itself is an ind-object in the category $\mathcal{P}$. On the other hand $H(\mathcal{M}_{\mathcal{P}})$ is a pro-object in $\mathcal{P}$.

Let $E \in End(\Psi)$ and E_M the corresponding endomorphism of $\Psi(M)$. There is a natural map $\varphi : H'(\mathcal{M}_{\mathcal{P}}) \to H(\mathcal{M}_{\mathcal{P}})^\vee$ given by

$$\varphi\Big((M; v_{\mathbb{Q}(0)}; f_B) \boxtimes B^\vee\Big)(E_M) := \quad f_B E_M v_{\mathbb{Q}(0)}(\mathbb{Q}(0));$$

Here we extended f_B to a morphism $\Psi(M) \to B$ postulating that its restriction to $gr^W_{-l}M$ for $l \neq n$ is zero. We will always extend the framing morphisms in a similar way.

Theorem 3.3 *a) $H'(\mathcal{M}_{\mathcal{P}})$ is dual to $H(\mathcal{M}_{\mathcal{P}})$.*

b) Δ provides $H'(\mathcal{M}_{\mathcal{P}})$ with a structure of a commutative Hopf algebra in the category $\mathcal{P}$ which is canonically isomorphic to $H(\mathcal{M}_{\mathcal{P}})^\vee$.

Proof. a) According to the Tannaka theory (which remains valid for fiber functors with values in the semisimple tensor categories) the fiber functor Ψ

provides an equivalence between the category $\mathcal{M}_{\mathcal{P}}$ and the category $H(\mathcal{M}_{\mathcal{P}})$-mod of finite dimensional $H(\mathcal{M}_{\mathcal{P}})$-modules in the category $\mathcal{P}$. So we will work with the category $H(\mathcal{M}_{\mathcal{P}})$-mod.

Let us show that φ is injective. Suppose that $\varphi\Big((M; v_{\mathbb{Q}(0)}; f_B) \boxtimes B^{\vee}\Big) = 0$. We may assume that $gr^W_p M = 0$ for $p > 0$ and $p < -n$ and $gr^W_0 M = \mathbb{Q}(0), gr^W_{-n} M = B$. Consider the cyclic submodule $M' := H(\mathcal{M}_{\mathcal{P}}) \cdot \Psi(\mathbb{Q}(0))$. Then its weights are bigger then $-n$, since otherwise $\varphi\Big((M; v_{\mathbb{Q}(0)}; f_B) \boxtimes B^{\vee}\Big) \neq 0$. So we get a morphism $M' \oplus B \to M$ which obviously respects the frames. On the other hand there is canonical projection $M' \oplus B \to \mathbb{Q}(0) \oplus B$ providing by the projection $M' \to gr^M_0 = \mathbb{Q}(0)$. Thus $(M; v_{\mathbb{Q}(0)}; f_B)$ is equivalent to $\mathbb{Q}(0) \oplus B$ (with the natural frame).

Now let us show that φ is surjective. Let $f \in H(\mathcal{M}_{\mathcal{P}})^{\vee}_B$ be a subobject isomorphic to $B^{\vee}$. Denote by $Kerf$ the corresponding subobject of $H(\mathcal{M}_{\mathcal{P}})_B$ such that $H(\mathcal{M}_{\mathcal{P}})_B / kerf = B$. Consider the $H(\mathcal{M}_{\mathcal{P}})$-module: $\mathbb{Q}(0) \oplus H(\mathcal{M}_{\mathcal{P}})_B / Kerf$ whith the action of $H(\mathcal{M}_{\mathcal{P}})$ provided by f, the nontrivial part of the action is a morphism $H(\mathcal{M}_{\mathcal{P}})_B \otimes \mathbb{Q}(0) \to H(\mathcal{M}_{\mathcal{P}})_B / Kerf$, and the obvious framing. The map φ sends it to f.

The part b) follows easyly from the definitions and the part a). The theorem is proved.

4 Conjectures on the Galois group of the category of mixed elliptic motives

1. A conjecture on Ext's in the category of mixed elliptic motives.

Lemma 4.1

$$RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), Sym^n \mathcal{H}(m)) = RHom_{\mathcal{MM}_{E^{(n)}}}(\mathbb{Q}(0), \mathbb{Q}(n+m))[n]_{sgn} \quad (15)$$

Proof. Let $p : E^{(n)} \longrightarrow Spec(k)$ be the canonical projection. Then we should have the motivic Leray spectral sequence

$$E_2^{p,q} = Ext^p_{\mathcal{MM}_k}\Big(\mathbb{Q}(0), R^q p_* \mathbb{Q}(n+m)\Big)$$

degenerating at E_2 and abbuting to $Ext^{p+q}_{\mathcal{MM}_{E^{(n)}}}\Big(\mathbb{Q}(0), \mathbb{Q}(n+m)\Big)$. Since

$$H^i(E^{(n)})_{sgn} = 0 \quad \text{for} \quad i \neq n; \qquad H^n(E^{(n)})_{sgn} = Sym^n h^1(E)$$

we get (15). Conjecture (3.1) tells us that the cohomology of the elliptic motivic complexes are given by the formula

$$R^i Hom_{\mathcal{MM}_k}(\mathbb{Q}(0), Sym^n \mathcal{H}(m)) \otimes \mathbb{Q} = gr^{\gamma}_{n+m} K_{n+2m-i}(E^{(n)})_{sgn} \otimes \mathbb{Q} \quad (16)$$

Let us choose a fiber functor $\varphi : \mathcal{P}_E \to Vect_{\mathbb{Q}}$. Set $H := \varphi(\mathcal{H})$ Then $\varphi(\mathbb{Q}(1))$ corresponds to the one-dimensional representation $det : GL_2(H) \to \mathbb{G}_m$ and $S^m\mathcal{H}(n)$ to the GL_2-module $S^m H(n)$, both with the trivial $LieU(\mathcal{M}_E)$-action.

The following conjecture plays a crucial role. Our results are little steps towards its confirmation.

Conjecture 4.2 *Let E be an elliptic curve over an algebraicaly closed field k The canonical morphism*

$$RHom_{\mathcal{M}_E}(\mathbb{Q}(0), S^n\mathcal{H}(m)) \longrightarrow RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m))$$

is a quasiisomorphism. In particular the Ext groups in the category $\mathcal{M}_E$ should be isomorphic to the Ext groups (between the same objects) in the category $\mathcal{MM}_k$.

Remark. This conjecture should not be true if k is not an algebraically closed field. A similar conjecture for the category of mixed Tate motives is expected to be valid for all fields k.

Question. Consider the mixed category $\mathcal{M}_{\{X_i\}}$ generated by given simple pure motives X_i. When could we expect that the Ext's in the category $\mathcal{M}_X$ coinside with the Ext's (between the same objects) in the category $\mathcal{MM}_k$?

2. The main hero: a Lie coalgebra $\mathcal{L}^*(E)$. Let us assume for simplicity that E is an ellliptic curve over a field k without complex multiplication. Then any simple object of the category $\mathcal{P}_E$ of pure elliptic motives is isomorphic to just one of the objects $S^n\mathcal{H}(m)$.

$\mathcal{P}_E$ is a rigid tensor category. Let us define a new tensor structure $\otimes'$ on this category setting

$$S^n\mathcal{H}(m) \otimes' S^{n'}\mathcal{H}(m') \stackrel{=}{\longrightarrow} S^{n+n'}\mathcal{H}(m+m')$$

where the morphism is the usual tensor product followed by the projection to the $S^{n+n'}\mathcal{H}(m+m')$ component. We get an abelian tensor category $\mathcal{P}_E^*$ which will be called *the small tensor category of elliptic motives.*

Remark. $\mathcal{P}_E^*$ is not a rigid tensor category since there is no dual object for $S^n\mathcal{H}(m)$ if $n > 0$.

The mixed Tate motives are contained in the category of mixed elliptic motives, and we have a canonical functor $\mathcal{M}_T(k) \longrightarrow \mathcal{M}_E$. Let $\pi^* : \mathcal{L}(k) \to \mathcal{L}(E)$ be the corresponding morphism of Lie coalgebras.

Conjecture 4.3 *There exists a Lie subcoalgebra $\mathcal{L}^*(E) \subset \mathcal{L}(E)$ which enjoys the following properties:*

1. $\mathcal{L}^*(E)$ is also a Lie coalgebra in the small tensor category $\mathcal{P}_E^*$.

2. The natural morphism of cohomology groups

$$H^\bullet_{\mathcal{P}_E^*}(\mathcal{L}^*(E)) \longrightarrow H^\bullet_{\mathcal{P}_E^*}(\mathcal{L}(E))$$

is an isomorphism.

3. π^* maps $\mathcal{L}(k)_{\mathbb{Q}(n)^\vee}$ isomorphically onto $\mathcal{L}^*(E)_{\mathbb{Q}(n)^\vee}$.

A construction of a Lie coalgebra in the tensor category $\mathcal{P}_E^*$ which hypothetically satisfies all these properties is given in s. 4.5.

Remarks. 1. A Lie subcoalgebra of $\mathcal{L}(E)$ is usually not a Lie coalgebra in the tensor category $\mathcal{P}_E^*$.

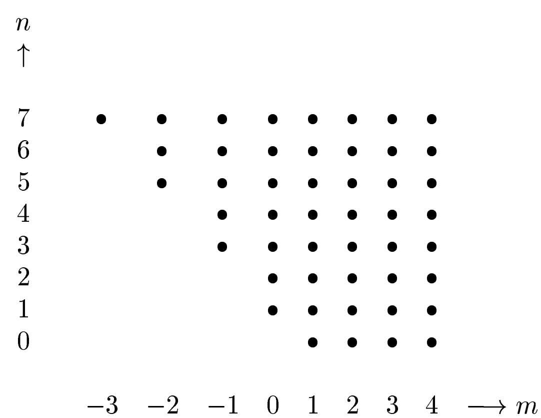
2. $H^\bullet_{\mathcal{P}_E^*}(\mathcal{L}^*(E))$ are quite different from $H^\bullet_{\mathcal{P}_E}(\mathcal{L}^*(E))$. The reason is clear from the following example:

$$(V \boxtimes \mathcal{H}) \otimes (V \boxtimes \mathcal{H}) = (V \otimes V) \boxtimes S^2\mathcal{H} \oplus (V \otimes V) \boxtimes S^2\mathcal{H}$$

$$(V \boxtimes \mathcal{H}) \otimes' (V \boxtimes \mathcal{H}) = (V \otimes V) \boxtimes S^2\mathcal{H}$$

3. Consider $\mathcal{L}^*(E)$ as a Lie coalgebra in $\mathcal{P}_E$. Let $\mathcal{M}_E^*$ be the category of comodules over it in the category $\mathcal{P}_E$. If $\mathcal{L}^*(E)$ is a Lie coalgebra of $\mathcal{L}(E)$, then $\mathcal{M}_E^*$ is a subcategory of $\mathcal{M}_E$.

The weight consideration shows that the picture of nonzero isotypical components of $\mathcal{L}^*(E)$ looks as follows:



A boldface point with coordinates (m, n) corresponds to nonzero isotypical component $\mathcal{L}^*(E)_{S^n\mathcal{H}(m)^\vee}$. For example $\mathcal{L}^*(E)_{\mathbb{Q}(0)} = 0$, $\mathcal{L}^*(E)_{S^2\mathcal{H}(-1)^\vee} = 0$.

3. Corollaries. According to conjectures (4.3) and (4.2) one has the following quasiisomorphisms:

$$\left(\Lambda^\bullet_{\mathcal{P}_E^*}\mathcal{L}^*(E), \partial\right)_{S^n\mathcal{H}(m)^\vee} \stackrel{4.3}{=} RHom_{\mathcal{M}_E}(\mathbb{Q}(0), S^n\mathcal{H}(m)) \stackrel{4.2}{=} \tag{17}$$

$$RHom_{\mathcal{M}\mathcal{M}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m)) \tag{18}$$

So we get the basic formula

$$H^i(\Lambda^\bullet_{\mathcal{P}_E^*}\mathcal{L}^*(E))_{S^n\mathcal{H}(m)^\vee} = gr^\gamma_{n+m} K_{n+2m-i}(E^{(n)})_{sgn} \otimes \mathbb{Q} \tag{19}$$

Using this we immediately conclude that

$$\mathcal{L}^*(E)_{\mathbb{Q}(1)^\vee} = k^* \boxtimes \mathbb{Q}(1)^\vee, \quad \mathcal{L}^*(E)_{\mathcal{H}^\vee} = J \boxtimes \mathcal{H}^\vee$$

More generally, let $J_{S^{2m+1}\mathcal{H}(-m)}$ be the image of the subgroup of homologically trivial cycles in $CH^m(E^{(2m-1)})_{sgn}$ under the Abel-Jacobi map. Then

$$\mathcal{L}^*(E)_{S^{2n+1}\mathcal{H}(-n)^\vee} = J_{S^{2n+1}\mathcal{H}(-n)} \boxtimes S^{2n+1}\mathcal{H}(-n)^\vee, \quad n > 0$$

Indeed, if H is a pure motive of weight 1 then

$$C^\bullet(\mathcal{L}^*(E))_H = \mathcal{L}^*(E)_H = H^1_{\mathcal{P}^*(E)}(\mathcal{L}^*(E))_H \stackrel{2}{=} H^1(\mathcal{L}^*(E))_H$$

Recall that $B_2(k)$ is the Bloch group of a field k. Let $B_3(k)$ be its analog for the classical trilogarithm introduced in [G1-2]).

Lemma 4.4 *(17) together with the results of [G2], [GL] implies that the lower left corner of the diagram looks as folows (all groups are tensored by $\mathbb{Q}$):*

$$\begin{array}{cccccc} J_{S^3\mathcal{H}(-1)} & \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet & \bullet \\ & J & B_3^*(E) & \bullet & \bullet & \bullet \\ & 0 & k^* & B_2(k) & B_3(k) & \bullet \end{array}$$

Proof. The structure of the bottom row follows from 3) and the main results in [G2]. The $\mathcal{H}(1)^\vee$-isotypical component of the standard complex of $\mathcal{L}^*(E)$ is

$$\mathcal{L}^*(E)_{\mathcal{H}(1)^\vee} \longrightarrow \mathcal{L}^*(E)_{\mathbb{Q}(1)^\vee} \otimes \mathcal{L}^*(E)_{\mathcal{H}^\vee} = k^* \otimes J \boxtimes \mathcal{H}(1)^\vee$$

The motivic interpretation of the group $B_3^*(E)$ as a group of $\mathcal{H}(1)$-framed mixed elliptic motives (see theorem 7.15 below) leads to a morphism of complexes

$$\begin{array}{ccc} B_3^*(E) \boxtimes \mathcal{H}(1)^\vee & \longrightarrow & k^* \otimes J \boxtimes \mathcal{H}(1)^\vee \\ \downarrow & & \downarrow = \\ \mathcal{L}^*(E)_{\mathcal{H}(1)^\vee} & \longrightarrow & k^* \otimes J \boxtimes \mathcal{H}(1)^\vee \end{array}$$

Thanks to the main result of [GL] and property 2) this map is a quasiisomorphism. So $\mathcal{L}^*(E)_{\mathcal{H}(1)^\vee} = B_3^*(E) \boxtimes \mathcal{H}(1)^\vee$.

The Lie coalgebra $\mathcal{L}(E)$ has a much more complicated structure, see s.4.8 below for the simplest example: description of $\mathcal{L}(E)_{\mathbb{Q}(1)^\vee}$.

Zero cycles on $E^{(n)}$. The only way how the object $S^n\mathcal{H}$ could appear as a direct summand of a tensor product of n simple objects of negative weights *in the small tensor category $\mathcal{P}_E^*$* is this:

$$S^n\mathcal{H} \quad = \quad \mathcal{H} \otimes' ... \otimes' \mathcal{H} \quad (n \text{ times})$$

Set

$$\mathcal{L}^*(E)_{S^2\mathcal{H}^\vee} := \quad B^*_{S^2\mathcal{H}} \boxtimes S^2\mathcal{H}^\vee \tag{20}$$

The $S^2\mathcal{H}^\vee$- isotypic component of the cochain complex $\Lambda^\bullet_{\mathcal{P}^*_E}\mathcal{L}^*(E)$ looks as follows:

$$\Big(\mathcal{L}^*(E) \longrightarrow \Lambda^2_{\mathcal{P}^*_E}\mathcal{L}^*(E)\Big)_{S^2\mathcal{H}^\vee} \quad = \quad \Big(B^*_{S^2\mathcal{H}}(E) \stackrel{\delta_{S^2\mathcal{H}}}{\longrightarrow} \Lambda^2 J\Big) \boxtimes S^2\mathcal{H}^\vee$$

Similarly the $S^n\mathcal{H}^\vee$-isotypic component of $\Lambda^\bullet_{\mathcal{P}^*_E}\mathcal{L}^*(E)$ is $(\quad ... \longrightarrow \Lambda^n J)\boxtimes S^n\mathcal{H}^\vee$. Therefore we get

$$Ext^n(\mathbb{Q}(0), S^n\mathcal{H}) \quad = \quad CH^n(E^{(n)})_{sgn} \quad = \quad \text{a quotient of} \quad \Lambda^n J \qquad (21)$$

The last equality indeed follows from the Bloch's theorem on zero cycles on abelian varieties [Bl5].

Similarly we deduce that for $m > 0$ one should have

$$Ext^{n+m}_{\mathcal{M}_E}(\mathbb{Q}(0), S^n\mathcal{H}(m)) \quad = \quad \text{a quotient of} \quad \Lambda^n J \otimes_{\mathbb{Q}} K^M_m(k)$$

4. The duality between DG Lie coalgebras and commutative DG algebras. Let $\mathcal{L}^\bullet$ be a DG Lie coalgebra in a tensor category $\mathcal{C}$, which is not supposed to be a rigid tensor category. This means that we have a complex $(\mathcal{L}^\bullet, \partial)$ and a cobracket

$$\delta : \mathcal{L}^\bullet \to \Lambda^2_{\mathcal{C}}(\mathcal{L}^\bullet)$$

which is a morphism of complexes satisfying the co Jacobi identity. Let

$$C(\mathcal{L}^\bullet) := \mathcal{L}^\bullet[-1] \oplus S^2_{\mathcal{C}}(\mathcal{L}^\bullet[-1]) \oplus S^3_{\mathcal{C}}(\mathcal{L}^\bullet[-1]) \oplus ...$$

be the free super commutative algebra (without unit) generated by $\mathcal{L}^\bullet[-1]$. There are two cohomological differentials on $C(\mathcal{L}^\bullet)$: the first is provided by the differential in $\mathcal{L}^\bullet$, and the second comes via the Leibniz rule from the cobracket δ. Their sum is a differential providing a structure of a commutative DG algebra on $C(\mathcal{L}^\bullet)$. For example, if $\mathcal{L}^\bullet = \mathcal{L}$ is concentrated in degree 0, then $S^n_{\mathcal{C}}(\mathcal{L}[-1]) = \Lambda^n_{\mathcal{C}}\mathcal{L}[-n]$ and we get the standard cochain complex of the Lie coalgebra:

$$C(\mathcal{L}) := \mathcal{L}[-1] \oplus \Lambda^2_{\mathcal{C}}(\mathcal{L})[-2] \oplus \Lambda^3_{\mathcal{C}}(\mathcal{L})[-3] \oplus ...$$

Let $\mathcal{L}^\bullet$ be a DG Lie coalgebra and $A^\bullet$ is a DG commutative algebra. Define $MC(Hom_k(\mathcal{L}^\bullet[-1], A^\bullet))$ as the set of all degree zero elements in $\omega \in Hom_k(\mathcal{L}^\bullet[-1], A^\bullet)$ satisfying the Maurer-Cartan equation $d\omega + \frac{1}{2}\omega \wedge \omega = 0$. Here $\omega \wedge \omega$ is the composition

$$\mathcal{L}^\bullet[-1] \to S^2_{\mathcal{C}}(\mathcal{L}^\bullet[-1]) \to S^2_{\mathcal{C}}A^\bullet \to A^\bullet$$

If we think of ω as of element in $(\mathcal{L}^\bullet[-1])^\vee \otimes A^\bullet$, we get the usual formula for $\omega \wedge \omega$. Then one has

$$MC(Hom_k(\mathcal{L}^\bullet[-1], A^\bullet)) \quad = \quad Hom_{DGCom}(C(\mathcal{L}), A)$$

We get a functor

$$C : DGCoLie \to DGCom$$

There is a functor in the opposite direction:

$$F : DGCom \to DGCoLie$$

satisfying

$$Hom_{DGcoLie}(\mathcal{L}^\bullet, F(A^\bullet)) \quad = \quad MC(Hom_k(\mathcal{L}^\bullet, A^\bullet[1]))$$

Namely, $F(A^\bullet)$ is the cofree Lie supercoalgebra $\mathcal{F}(A^\bullet[1])$ generated by the complex $A^\bullet[1]$. There is a canonical projection $p : \mathcal{F}(A^\bullet[1]) \longrightarrow A^\bullet[1]$. According to the universality property of the cofree Lie coalgebras to define the differential on $\mathcal{F}(A^\bullet[1])$ it is sufficient to define its projection $\mathcal{F}(A^\bullet[1]) \to A^\bullet[1]$. By definition it is the sum of the differential in $A^\bullet[1]$ and the product $\Lambda^2(A^\bullet[1]) = S^2(A^\bullet)[2] \to A^\bullet[1]$.

The functors C and F are obviously adjoint. So there are canonical morphisms

$$A^\bullet \longrightarrow C \circ F(A^\bullet) \qquad \mathcal{L}^\bullet \longrightarrow F \circ C(\mathcal{L}^\bullet)$$

Theorem 4.5 *These morphisms are quasiisomorphisms.*

This theorem was proved by Quillen [Q] when $\mathcal{C}$ is the category of $\mathbb{Q}$-vector spaces, but it is true for an arbitrary tensor category $\mathcal{C}$.

The functor F has another description via the bar construction. Namely, let $B(A^\bullet)$ be the reduced bar complex of a DG commutative algebra $A^\bullet$. It has a structure of a Hopf algebra. Let $B_0(A^\bullet)$ be the augmentation ideal of $B(A^\bullet)$. Then the space $B_0(A^\bullet)/(B_0(A^\bullet) \cdot B_0(A^\bullet))$ of the indecomposable elements has a natural structure of a DG Lie coalgebra (a good reference for the Bar construction is ch. 2 of [BK]).

5. A cycle construction of $\mathcal{L}^*(E)$ (Compare with [Bl4]). Let sgn be the one dimensional alternating representation of G_n where a generator of each factor $\mathbb{Z}/2\mathbb{Z}$ acts by -1 and the restriction to S_n is the alternating representation. The group G_n acts naturally on E^n.

The idea of the construction. Let $\Gamma_X(n)$ be a motivic complex on a regular scheme X, i.e., it is a *complex* of $\mathbb{Q}$-vector spaces quasiisomorphic to $RHom_{\mathcal{MM}_X}(\mathbb{Q}(0), \mathbb{Q}(n))$. We will need also a canonical morphism of complexes $\Gamma_X(m) \times \Gamma_Y(n) \to \Gamma_{X \times Y}(m+n)$. We will take below the complex of Bloch's Higher Chow groups on X as a concrete version of $\Gamma_X(n)$ to work with. Then

$$\mathcal{N}^*(E) := \oplus_{2m+n>0, n\geq 0} \Gamma_{E^n}(m+n)_{sgn} \boxtimes S^n\mathcal{H}(m)^\vee$$

has a natural structure of a commutative DG algebra (without unit) in the small tensor category of pure elliptic motives. Namely, the product is provided by the natural morphism of complexes

$$\Gamma_{E^n}(m+n)_{sgn} \times \Gamma_{E^{n'}}(m'+n')_{sgn} \quad \longrightarrow \quad \Gamma_{E^{n+n'}}(m+m'+n+n')_{sgn}$$

(take the external product of the complexes on $E^{n+n'}$ and project it onto the signum part with respect to the action of the group $G_{n+n'}$). It remains to apply the functor F!

Now let us spell out the details for the complexes of Bloch's Higher Chow groups. Let $\mathcal{Z}^n(E^m, k)$ be the $\mathbb{Q}$-vector space of codimension n cycles with $\mathbb{Q}$ coefficients on $E^m \times (\mathbb{P}^1 \backslash \{1\})^k$ which are skewsymmetric with respect to the action of G_k on $(\mathbb{P}^1 \backslash \{1\})^k$ and meet the faces of the coordinate cube defined by setting some of the coordinates equal to 0 or ∞ properly. Set $\mathcal{Z}^{m+n}(E^n, c)_{sgn} := (\mathcal{Z}^{m+n}(E^n, c) \otimes sgn)^{G_n}$. We define a complex $\mathcal{Z}^{m+n}(E^n, *)_{sgn}$ placing

$$\mathcal{Z}^{m+n}(E^n, c)_{sgn} \quad \text{in degree} \quad 2n + m - c$$

The differential is the alternating sum of the face maps:

$$\mathcal{Z}^{m+n}(E^n, c)_{sgn} \longrightarrow \mathcal{Z}^{m+n}(E^n, c-1)_{sgn}$$

A $\otimes'$-product of $RHom$'s

$$RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m)) \otimes' RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^{n'}\mathcal{H}(m')) \longrightarrow \tag{22}$$

$$RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^{n+n'}\mathcal{H}(m+m'))$$

is provided by the usual tensor product

$$RHom(\mathbb{Q}, A) \otimes RHom(\mathbb{Q}, B) \longrightarrow RHom(\mathbb{Q}, A \otimes B)$$

and the canonical projection $S^n\mathcal{H}(m) \otimes S^{n'}\mathcal{H}(m') \to S^{n+n'}\mathcal{H}(m+m')$.

Lemma 4.6 *a) One has a quasiisomorphism*

$$\mathcal{Z}^{m+n}(E^n, *)_{sgn} \quad = \quad RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m))$$

b) The product of cycles followed by the projection to the $G_{n+n'}$-alternating part provides a morphism of complexes

$$\mathcal{Z}^{m+n}(E^n, *)_{sgn} \otimes \mathcal{Z}^{m'+n'}(E^{n'}, *')_{sgn} \longrightarrow \mathcal{Z}^{m+n+m'+n'}(E^{n+n'}, * + *')_{sgn}$$

which coincides in the derived category with the $\otimes'$-product of RHom's (22).

Proof. Follows from lemma (4.1) and the results of Bloch [Bl2], [BK].
Set

$$\mathcal{N}^*(E) := \oplus_{2m+n>0} \mathcal{Z}^{m+n}(E^n, *)_{sgn} \boxtimes S^n\mathcal{H}(m)^\vee$$

There is a commutative product on $\mathcal{N}^*(E)$ given by

$$\Big(\mathcal{Z}^{m+n}(E^n, *)_{sgn} \boxtimes S^n\mathcal{H}(m)^\vee\Big) \otimes \Big(\mathcal{Z}^{m'+n'}(E^{n'}, *')_{sgn} \boxtimes S^{n'}\mathcal{H}(m')^\vee\Big)$$

$$\Big(\mathcal{Z}^{m+n+m'+n'}(E^{n+n'}, * + *')_{sgn} \boxtimes S^{n+n'}\mathcal{H}(m+m')^\vee\Big)$$

Proposition 4.7 *$\mathcal{N}^*(E)$ is a commutative DG algebra (without unit) in the small tensor category of pure elliptic motives.*

Thus setting $\tilde{\mathcal{L}}^*(E) := F\mathcal{N}^*(E)$ and using theorem 4.5 we get a proof of theorem 1.1.

Conjecture 4.8 *$H^i(F\mathcal{N}^*(E)) = 0$ if $i \neq 0$.*

$H^0(F\mathcal{N}^*(E))$ is a Lie coalgebra in the tensor category $\mathcal{P}_E^*$ which is our candidate for $\mathcal{L}^*(E)$.

Proposition 4.9 *Assuming the conjecture (4.8) one has for $(n,m) \neq (0,0)$:*

$$\left(H^*_{\mathcal{P}^*_E}(H^0(\tilde{\mathcal{L}}^*(E)))\right)_{S^n\mathcal{H}(m)^\vee} = RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m)) \boxtimes S^n\mathcal{H}(m)^\vee$$

Proof. This folows immediately from theorem 4.5 and lemma 4.6.

6. The Freeness conjecture for mixed elliptic motives. The Lie coalgebras in this paper are Ind-objects in the cateogory of pure motives. Denote by $L(E)$, $L^*(E)$, $I^*(E)$ the Lie algebras dual to the Lie coalgebras $\mathcal{L}(E)$, $\mathcal{L}^*(E)$, $\mathcal{I}^*(E)$. Set

$$I^*(E) := \oplus_{n+m>1} L^*(E)_{S^n\mathcal{H}(m)}$$

It is clear from the picture above that $I^*(E)$ is an ideal in $L^*(E)$ and

$$L^*(E)/I^*(E) = (k^*)^\vee \boxtimes \mathbb{Q}(1) \oplus J^\vee \boxtimes \mathcal{H} \tag{23}$$

is an abelian Lie algebra.

Conjecture 4.10 *$I^*(E)$ is a free Lie algebra in the tensor category $\mathcal{P}_E^*$.*

A Lie algebra L in $\mathcal{P}_E^*$ is free if and only if $H^i_{\mathcal{P}^*_E}(L) = 0$ for $i > 1$.

Remark. According to the property 3) this conjecture implies the Freeness conjecture for mixed Tate motives.

7. The vector space $\mathcal{L}(E)_{\mathbb{Q}(1)^\vee}$. It follows from 19 that one should have

$$\mathcal{L}(E)_{S^{2n+1}\mathcal{H}(-n)^\vee} = J_{S^{2n+1}\mathcal{H}(-n)} = CH^{n+1}(E^{2n+1})_{sgn}$$

Let M be a simple object of $\mathcal{P}_E$. According to H.Weyl's theorem $Hom_{\mathcal{P}_E}(M, \otimes^m\mathcal{H})$ is an irreducible G_m-module. Denote it by $\rho_M^{(m)}$ (we will omit (m) sometimes). Set

$$CH^{2n+2}(E^{4n+2})_{\rho_{\mathbb{Q}(1)^\vee}} := \left(CH^{2n+2}(E^{4n+2}) \otimes \rho_{\mathbb{Q}(1)^\vee}\right)_{G_{4n+2}} \subset S^2(J_{S^{2n+1}\mathcal{H}(-n)})$$

For any integer $n \geq 0$ one can define an abelian group $B^{(n)}_{\mathbb{Q}(1)}$ together with the following exact sequence:

$$0 \longrightarrow k^*_{\mathbb{Q}} \longrightarrow B^{(n)}_{\mathbb{Q}(1)} \longrightarrow CH^{2n+2}(E^{4n+2})_{\rho_{\mathbb{Q}(1)^\vee}} \longrightarrow 0 \tag{24}$$

For $n = 0$ this is the group $B_2(E)$ we discussed in chapter 2. In general the extension 24 comes in a similar way from the biextension related to codimension $n+1$ cycles in E^{2n+1} (see [Bl3]).

Let us take the sum of all these extensions:

$$0 \longrightarrow k^* \otimes \mathbb{Q}[t] \longrightarrow \oplus_{n\geq 0} B^{(n)}_{\mathbb{Q}(1)} \longrightarrow \oplus_{n\geq 0} CH^{2n+2}(E^{4n+2})_{\rho_{\mathbb{Q}(1)^\vee}} \boxtimes \mathbb{Q}(1)^\vee \longrightarrow 0$$

There is a homomorphism

$$\alpha : k^* \otimes \mathbb{Q}[t] \longrightarrow k^*, \quad a \otimes t^n \longmapsto a$$

Theorem 4.11

$$\mathcal{L}_{\mathbb{Q}(1)^\vee} = \frac{\oplus_{n\geq 0} B^{(n)}_{\mathbb{Q}(1)}}{ker\alpha}$$

Using the identification

$$\oplus_{n\geq 0}\Big(\Lambda^2 \mathcal{L}(E)_{S^{2n+1}\mathcal{H}(-n)^\vee}\Big)_{\mathbb{Q}(1)^\vee} = \oplus_{n\geq 0} CH^{2n+2}(E^{4n+2})_{\rho_{\mathbb{Q}(1)^\vee}} \boxtimes \mathbb{Q}(1)^\vee$$

we get a canonical homomorphism, provided by (24)

$$\mathcal{L}(E)_{\mathbb{Q}(1)^\vee} \longrightarrow \oplus_{n\geq 0}\Big(\Lambda^2 \mathcal{L}(E)_{S^{2n+1}\mathcal{H}(-n)^\vee}\Big)_{\mathbb{Q}(1)^\vee}$$

This homomorphism gives the cobracket δ. Its kernel is isomorphic to k^*.

Notice how simple is $\mathcal{L}^*(E)_{\mathbb{Q}(1)^\vee} = k^*_{\mathbb{Q}}$ and how complicated is $\mathcal{L}(E)_{\mathbb{Q}(1)^\vee}$!

8. Reflections on elliptic motivic complexes. Suppose that E is defined over an algebraically closed field k. I conjecture that:

a) For any m, n such that $n \geq 0, n + 2m > 0$ there exists an abelian group $B^*_{S^n\mathcal{H}(m)}$ together with homomorphisms

$$\delta_1 : B^*_{S^n\mathcal{H}(m)} \longrightarrow B^*_{S^{n-1}\mathcal{H}(m)} \otimes J, \quad n + 2m > 0$$

$$\delta_2 : B^*_{S^n\mathcal{H}(m)} \longrightarrow B^*_{S^n\mathcal{H}(m-1)} \otimes k^* \quad m \geq 1$$

For $m \geq 1$ one should have

$$B^*_{S^{2m+1}\mathcal{H}(-m)} = J_{S^{2m+1}\mathcal{H}(-m)} \quad \text{and} \quad B^*_{\mathbb{Q}(m)} = B_m(k)$$

where $B_m(k)$ are the groups introduced in [G1-2]. For instance $B^*_{\mathcal{H}} = J$ and $B^*_{\mathbb{Q}(1)} = k^*$.

b) One should get a bicomplex $\Gamma^*_{S^n\mathcal{H}(m)}$ of the following shape.

For $m > 0$:

$$\begin{array}{ccccc}
B^*_{S^n\mathcal{H}(m)} & \longrightarrow & B^*_{S^n\mathcal{H}(m-1)} \wedge k^* & \longrightarrow ... \longrightarrow & B^*_{S^n\mathcal{H}(1)} \wedge \Lambda^{m-1}k^* \\
\downarrow & & \downarrow & ... & \downarrow \\
B^*_{S^{n-1}\mathcal{H}(m)} \wedge J & \longrightarrow & B^*_{S^{n-1}\mathcal{H}(m-1)} \wedge J \wedge k^* & \longrightarrow ... \longrightarrow & B^*_{S^{n-1}\mathcal{H}(1)} \wedge J \wedge \Lambda^{m-1}k^* \\
\downarrow & & \downarrow & ... & \downarrow \\
... & & ... & ... & ... \\
\downarrow & & \downarrow & ... & \downarrow \\
B^*_{\mathbb{Q}(m)} \wedge \Lambda^n J & \longrightarrow & B^*_{\mathbb{Q}(m-1)} \wedge \Lambda^n J \wedge k^* & \longrightarrow ... \longrightarrow & \Lambda^n J \wedge \Lambda^m k^*
\end{array}$$

Here the horizontal differentials are $b \wedge x \longmapsto \delta_1(b) \wedge x$ and the vertical ones are $b \wedge x \longmapsto \delta_2(b) \wedge x$.

For $m = 0$:

$$B^*_{S^n\mathcal{H}} \longrightarrow B^*_{S^{n-1}\mathcal{H}} \wedge J \longrightarrow ... \longrightarrow B^*_{S^2\mathcal{H}} \wedge \Lambda^{n-2}J \longrightarrow \Lambda^n J$$

For $m < 0$, set $m' := -m$. Then

$$B^*_{S^n\mathcal{H}(-m')} \longrightarrow B^*_{S^{n-1}\mathcal{H}(-m')} \otimes J \longrightarrow ... \longrightarrow B^*_{S^{2m'+1}\mathcal{H}(-m')} \otimes \Lambda^{n-2m'-1}J$$

c) The total complex associated with the bicomplex $\Gamma^*_{S^n\mathcal{H}(m)}$ (abusing notations I will denote it also by $\Gamma^*_{S^n\mathcal{H}(m)}$) after tensoring with $\mathbb{Q}$ should be quasi-isomorphic to $RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(m))$.

Example. $\Gamma^*_{\mathcal{H}(m)}$ is the total complex associated with the bicomplex

$$\begin{array}{ccccccc}
B^*_{\mathcal{H}(m)} & \to & B^*_{\mathcal{H}(m-1)} \wedge k^* & \to ... \to & B^*_{\mathcal{H}(2)} \wedge \Lambda^{m-2}k^* & \to & B^*_3(E) \wedge \Lambda^{m-1}k^* \\
\downarrow & & \downarrow & ... & \downarrow & & \downarrow \\
B^*_{\mathbb{Q}(m)} \wedge J & \to & B^*_{\mathbb{Q}(m-1)} \wedge J \wedge k^* & \to ... \to & B^*_{\mathbb{Q}(2)} \wedge J \wedge \Lambda^{m-2}k^* & \to & J \wedge \Lambda^m k^*
\end{array}$$

d) There should be a variation $P_{n,m}$ of *mixed elliptic motives* framed by $\mathbb{Q}(0)$ and $S^n\mathcal{H}(m)$ over a certain *finite dimensional* variety $X(n,m)$ over k. The groups $B^*_{S^n\mathcal{H}(m)}$ should come from it as follows. The variation $P_{m,n}$ provides a homomorphism

$$\tilde{l}_{n,m} : \mathbb{Q}[X(n,m)(k)] \to \mathcal{A}(\mathbb{Q}(0), S^n\mathcal{H}(m))$$

where $l_{n,m}(\{x\})$ for $x \in X_{n,m}$ is the class of the framed mixed motive $P_{n,m}(x)$. ($P_{n,m}(x)$ is the fiber at x of the variation $P_{n,m}$). Passing to the coalgebra Lie we get a homomorphism

$$l_{n,m} : \mathbb{Q}[X(n,m)(k)] \to \mathcal{L}(E)_{S^n\mathcal{H}(m)^\vee}$$

Set $B_{S^n\mathcal{H}(m)} := Im(l_{n,m})$. The group $B^*_{S^n\mathcal{H}(m)}$ is the largest $\mathbb{Q}$-subspace of $B_{S^n\mathcal{H}(m)}$ such that the restriction of the coproduct δ to the group $B^*_{S^n\mathcal{H}(m)}$ has non zero components only in

$$\mathcal{L}_{S^{n-1}\mathcal{H}(m)^\vee} \wedge J \boxtimes \mathcal{H}^\vee \quad \oplus \quad \mathcal{L}_{S^n\mathcal{H}(m-1)^\vee} \wedge k^* \boxtimes \mathbb{Q}(1)^\vee, \quad \text{if } m > 1$$

and in

$$\mathcal{L}_{S^{n-1}\mathcal{H}(m)^\vee} \wedge J \boxtimes \mathcal{H}^\vee, \quad \text{if } m \leq 1$$

By definition δ_1 and δ_2 are the components of the restriction of δ to $B^*_{S^n\mathcal{H}(m)} \boxtimes S^n\mathcal{H}(m)^\vee$. The restriction of δ to $B_{S^n\mathcal{H}(m)} \boxtimes S^n\mathcal{H}(m)^\vee$ may be more complicated.

The periods of the $\mathbb{R}$-Hodge realization of the variation $P_{n,m}$ are the new transcendental functions denoted $\mathcal{P}_{n,m}$ which is needed to get the special values $L(S^n\mathcal{H}, m+n)$.

Example. $P_{n,1}$ is the (motivic) elliptic polylogarithm sheaf on $E\backslash 0$, and $\mathcal{P}_{n,1}$ are the Eisenstein-Kronecker series from 1.5. The group $B_{S^n\mathcal{H}(1)}$ (resp. $B^*_{S^n\mathcal{H}(1)}$) should coincide with the group

$$\mathcal{B}_{n+2}(E) = \frac{\mathbb{Q}[E(k]}{\mathcal{R}_{n+2}(E)}, \quad \mathcal{B}^*_{n+2}(E) = \frac{I_E^{2n+2}}{\mathcal{R}_{n+2}(E)}$$

which will be defined later on. Here I_E is the augmentation ideal of the group algebra $\mathbb{Q}[E(k)]$.

8. The structure of $L^*(E)$ and elliptic motivic complexes. Let us define the $\mathbb{Q}$-vector spaces $C^*_{S^n\mathcal{H}(m)}$ by setting

$$H^1(I^*(E)) = \oplus C^*_{S^n\mathcal{H}(m)} \boxtimes S^n\mathcal{H}(m)^\vee$$

Let δ^* be the the cobracket in the Lie coalgebra $\mathcal{L}^*(E)$. Conjecture (4.10) means that

$$\delta^* : C^*_{S^n\mathcal{H}(m)} \longrightarrow C^*_{S^n\mathcal{H}(m-1)} \wedge k^* \quad \oplus \quad C^*_{S^{n-1}\mathcal{H}(m)} \wedge J \tag{25}$$

Lemma 4.12 *The $S^n\mathcal{H}(m)^\vee$-isotypical component of the Serre-Hochshild spectral sequence for the ideal $I^*(E) \subset L^*(E)$ computing cohomology of $L^*(E)$ collapces to the total complex associated with a bicomplex (which should be tensored $\boxtimes S^n\mathcal{H}(m)^\vee$):*

$$\begin{array}{ccccc}
C^*_{S^n\mathcal{H}(m)} & \rightarrow & C^*_{S^n\mathcal{H}(m-1)} \otimes k^* & \rightarrow \dots \rightarrow & C^*_{S^n\mathcal{H}(-[\frac{n-1}{2}])} \otimes S^{m+[\frac{n-1}{2}]}k^* \\
\downarrow & & \downarrow & & \downarrow \\
C^*_{S^{n-1}\mathcal{H}(m)} \otimes J & \rightarrow & C^*_{S^{n-1}\mathcal{H}(m)} \otimes J \otimes k^* & \rightarrow \dots \rightarrow & C^*_{S^{n-1}\mathcal{H}(-[\frac{n-2}{2}])} \otimes J \otimes S^{m+[\frac{n-2}{2}]}k^* \\
\downarrow & & \downarrow & & \downarrow \\
\dots & & \dots & & \dots \\
\downarrow & & \downarrow & & \downarrow \\
C^*_{\mathcal{H}(m)} \otimes \Lambda^{n-1}J & \rightarrow & C^*_{\mathcal{H}(m-1)} \otimes \Lambda^{n-1}J \otimes k^* & \rightarrow \dots \rightarrow & C^*_{\mathcal{H}(1)} \otimes \Lambda^{n-1}J \otimes \Lambda^{m-1}k^* \\
\downarrow & & \downarrow & & \downarrow \\
C^*_{\mathbb{Q}(m)} \otimes \Lambda^n J & \rightarrow & C^*_{\mathbb{Q}(m-1)} \otimes \Lambda^n J \otimes k^* & \rightarrow \dots \rightarrow & \Lambda^n J \otimes \Lambda^m k^*
\end{array}$$

The length of the rows decreases when we are going down. Here is how it looks for $n = 4, m = 1$:

$$\begin{array}{ccccc}
C^*_{S^4\mathcal{H}(1)} & \to & C^*_{S^4\mathcal{H}} \otimes k^* & \to & C^*_{S^4\mathcal{H}(-1)} \otimes \Lambda^2 k^* \\
\downarrow & & \downarrow & & \downarrow \\
C^*_{S^3\mathcal{H}(1)} \otimes J & \to & C^*_{S^3\mathcal{H}} \otimes J \otimes k^* & \to & C^*_{S^3\mathcal{H}(-1)} \otimes J \otimes \Lambda^2 k^* \\
\downarrow & & \downarrow & & \\
C^*_{S^2\mathcal{H}(1)} \otimes \Lambda^2 J & \to & C^*_{S^2\mathcal{H}} \otimes \Lambda^2 J \otimes k^* & & \\
\downarrow & & & & \\
C^*_{\mathcal{H}(1)} \otimes \Lambda^3 J & & & & \\
\downarrow & & & & \\
k^* \otimes \Lambda^2 J & & & &
\end{array}$$

Proof. The E_1 term of this spectral sequence is

$$E_1^{p,q} \quad = \quad C^p\Big(k^* \oplus J, \quad H^q_{\mathcal{P}^*_E}(\mathcal{I}^*(E))\Big)_{S^n\mathcal{H}(m)^\vee}$$

Since $H^q_{\mathcal{P}^*_E}(\mathcal{I}^*(E))$ is zero for $q > 1$ the only non zero rows are $E_1^{\bullet,0}$ and $E_1^{\bullet,1}$. Moreover $E_1^{p,0}$ is zero unless $p = n + m$, and

$$E_1^{m+n,0} = \Lambda^n J \otimes \Lambda^m k^*$$

Further, $E_1^{p,1}$ is non zero only if $0 \leq p < m + n$. If so, then

$$E_1^{p,1} \quad = \quad \oplus_{a+b=p} \quad C^*_{S^{n-a}\mathcal{H}(m-b)} \otimes \Lambda^a J \otimes \Lambda^b k^*$$

So the E_1 term of the spectral sequence gives us precisely the groups in the bicomplex. The differential $d_1 : E_1 \to E_1$ provides all the differentials exept the last two in the right bottom corner targeting to $\Lambda^n J \otimes \Lambda^m k^*$. These two we get from the differential d_2. The lemma is proved.

The structure of the quotient $L^*(E)/[I^*(E), I^*(E)]$. There is an exact sequence of Lie algebras

$$0 \to \frac{I^*(E)}{[I^*(E), I^*(E)]} \to \frac{L^*(E)}{[I^*(E), I^*(E)]} \to \frac{L^*(E)}{I^*(E)} \to 0 \qquad (26)$$

The action of $L^*(E)$ on $I^*(E)$ leads to the action of $L^*(E)/I^*(E)$ on $I^*(E)/[I^*(E), I^*(E)]$. The Lie algebra structure on $L^*(E)/[I^*(E), I^*(E)]$ is determined by this action.

The inclusion $L^*(E)_{\mathbb{Q}(1)^\vee} \oplus L^*(E)_{\mathcal{H}(1)^\vee} \hookrightarrow L^*(E)$ provides a canonical splitting $s : L^*(E)/I^*(E) \to L^*(E)/[I^*(E), I^*(E)]$ as an extension of $\mathbb{Q}$-vector spaces. Recall that $L^*(E)/I^*(E) = (k^*_{\mathbb{Q}})^\vee \boxtimes \mathbb{Q}(1) \oplus J^\vee_{\mathbb{Q}} \boxtimes \mathcal{H}$ is an abelian Lie algebra. So to define the Lie algebra structure on $L^*(E)/[I^*(E), I^*(E)]$ we need to know the $\mathbb{Q}$-vector spaces $C^*_{S^n\mathcal{H}(m)}$ and the homomorphisms (25). That is precisely what the following two conjectures are doing.

Conjecture 4.13 *i) There is a canonical map* $B^*_{S^n\mathcal{H}(m)} \hookrightarrow C^*_{S^n\mathcal{H}(m)}$ *such that* $\delta^*|_{B^*_{S^n\mathcal{H}(m)}} = \delta_1 + \delta_2$.

ii) $B^*_{S^n\mathcal{H}(m)} = C^*_{S^n\mathcal{H}(m)}$ *if* $n = 0, 1$. *Otherwise one has an exact sequence*

$$0 \to B^*_{S^n\mathcal{H}(m)} \to C^*_{S^n\mathcal{H}(m)} \to B^*_{S^n\mathcal{H}(-[\frac{n-1}{2}])} \otimes S^{m+[\frac{n-1}{2}]}k^* \to 0 \qquad (27)$$

Set $\bar{C}^*_{S^n\mathcal{H}(m)} := C^*_{S^n\mathcal{H}(m)}/B^*_{S^n\mathcal{H}(m)}$. Then δ^* induces a map

$$\bar{C}^*_{S^n\mathcal{H}(m)} \longrightarrow \bar{C}^*_{S^n\mathcal{H}(m-1)} \otimes k^* \quad \oplus \quad \bar{C}^*_{S^{n-1}\mathcal{H}(m)} \otimes J \qquad (28)$$

Now we can formulate the second part of conjecture (4.10).

Conjecture 4.14 *The second component of the map (28) is zero, and the first component coinsides with the homomorphism*

$$B^*_{S^n\mathcal{H}(-[\frac{n-1}{2}])} \otimes S^{m+[\frac{n-1}{2}]}k^* \longrightarrow B^*_{S^n\mathcal{H}(-[\frac{n-1}{2}])} \otimes S^{m+[\frac{n-1}{2}]-1}k^* \otimes k^* \qquad (29)$$

given by the identity $\times$ *the Koszul differential.*

Theorem 4.15 *Assume all the conjectures of this section. Then the complex* $\Gamma^*_{S^n\mathcal{H}(m)}$ *is quasiisomorphic to* $RHom_{\mathcal{M}_E}(\mathbb{Q}(0), S^n\mathcal{H}(m))$

Proof. The complex $RHom_{\mathcal{M}_E}(\mathbb{Q}(0), S^n\mathcal{H}(m))$ is quasiisomorphic to the standard complex $C^\bullet(\mathcal{L}(E))$ of the Lie coalgebra $\mathcal{L}(E)$. According to the property 1) there is a morphism of complexes

$$C^\bullet_{\mathcal{P}^*_E}(\mathcal{L}^*(E)) \longrightarrow C^\bullet_{\mathcal{P}_E}(\mathcal{L}(E))$$

which is a quasiisomorphism thanks to the property 2).

The part ii) of conjecture (4.13) just means that there is a canonical embedding of complexes

$$j : \Gamma^*_{S^n\mathcal{H}(m)} \hookrightarrow C^\bullet_{\mathcal{P}^*_E}(\mathcal{L}^*(E))$$

We are going to show that it is a quasiisomorphism.

Using the part i) of conjecture (4.13) we see that the quotient of the bicomplex we got from the spectral sequence along the bicomplex $\Gamma^*_{S^n\mathcal{H}(m)}$ looks as follows. The two bottom rows become zero, and each of the remaining rows is $B^*_?$ times a Koszul complex. For instance in the case $n = 4, m = 1$ the quotient is get

$$\begin{array}{ccccc}
B^*_{S^4\mathcal{H}(-1)} \otimes S^2k^* & \to & B^*_{S^4\mathcal{H}(-1)} \otimes k^* \otimes k^* & \to & B^*_{S^3\mathcal{H}(-1)} \otimes \Lambda^2 k^* \\
0\downarrow & & 0\downarrow & & 0\downarrow \\
B^*_{S^3\mathcal{H}(-1)} \otimes J \otimes S^2k^* & \to & B^*_{S^3\mathcal{H}(-1)} \otimes J \otimes k^* \otimes k^* & \to & B^*_{S^3\mathcal{H}(-1)} \otimes J \otimes \Lambda^2 k^* \\
0\downarrow & & 0\downarrow & & \\
B^*_{S^2\mathcal{H}} \otimes \Lambda^2 J \otimes k^* & \to & B^*_{S^2\mathcal{H}} \otimes \Lambda^2 J \otimes k^* & &
\end{array}$$

5 The complexes $B(E,n)^\bullet$ and $B^*(E,n)^\bullet$

1. An auxillary complex. Let A be an abelian group. Consider the following complex

$$\mathbb{Q}[A] \stackrel{\delta}{\longrightarrow} \mathbb{Q}[A] \otimes A_{\mathbb{Q}} \stackrel{\delta}{\longrightarrow} \mathbb{Q}[A] \otimes \Lambda^2 A_{\mathbb{Q}} \stackrel{\delta}{\longrightarrow} \mathbb{Q}[A] \otimes \Lambda^3 A_{\mathbb{Q}} \stackrel{\delta}{\longrightarrow} ... \tag{30}$$

The differential is defined by the formula

$$\delta : \{a\} \otimes b_1 \wedge b_2 \wedge ... \wedge b_m \longmapsto \{a\} \otimes a \wedge b_1 \wedge b_2 \wedge ... \wedge b_m \tag{31}$$

It is infinite if $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$ is infinite dimensional. Let I_A^k be the k-th degree of the augmentation ideal $I_A \subset \mathbb{Z}[A]$.

Lemma 5.1 $\delta(I_A^k) \subset I_A^{(k-1)} \otimes A$.

Proof. I_A^k is generated by the elements

$$(\{a_1\} - \{0\}) * (\{a_1\} - \{0\}) * ... * (\{a_k\} - \{0\})$$

Clearly

$$\delta(\{a_1\} - \{0\}) * ... * (\{a_k\} - \{0\}) = \sum_i \prod_{j \neq i} (\{a_j\} - \{0\}) * \{a_i\} \otimes a_i$$

So the complex (30) has the "diagonal" filtration by subcomplexes

$$I_A^n \stackrel{\delta}{\longrightarrow} I_A^{n-1} \otimes A_{\mathbb{Q}} \stackrel{\delta}{\longrightarrow} I_A^{n-2} \otimes \Lambda^2 A_{\mathbb{Q}} \stackrel{\delta}{\longrightarrow} I_A^{n-3} \otimes \Lambda^3 A_{\mathbb{Q}} \stackrel{\delta}{\longrightarrow} ...$$

Each graded quotient is isomorphic to the Koszul complex

$$S^n A_{\mathbb{Q}} \longrightarrow S^{n-1} A_{\mathbb{Q}} \otimes A_{\mathbb{Q}} \longrightarrow ... \longrightarrow A_{\mathbb{Q}} \otimes \Lambda^{n-1} A_{\mathbb{Q}} \longrightarrow \Lambda^m A_{\mathbb{Q}}$$

2. The groups $B_n(E)$. Recall that $B_0(E) := \mathbb{Z}$ and $B_1(E) = J(k)$. We will usually write J for $J(k)$. The group $B_2(E)$ is the group discussed in s. 2.1. In particular if $k = \bar{k}$ it is a quotient of $\mathbb{Z}[E(k)]$.

Recall that

$$E^{(n-1)} = \{(x_1, ..., x_n) \subset E^n | \sum x_i = 0\}$$

The group S_n acts naturally on $E^{(n-1)}$. Let $p_i : E^{(n-1)} \to E$ be the projection to the i-th factor. We will use the "coordinate notations" denoting $p_i^* f$ by $f(x_i)$

etc. Let us define the following diagram:

$$\begin{array}{ccc} & & K_n^M\Big(k(E^{(n-1)})\Big) \\ & {\scriptstyle \bar{\mu}_n} \nearrow & \\ S^n\Big(k(E)^*\Big) & & \\ & {\scriptstyle \beta_n} \searrow & \\ & & I_E^{2n} \end{array}$$

by setting

$$\bar{\mu}_n : f_1 \circ ... \circ f_n \longmapsto \sum_{\sigma \in S_n} (-1)^{|\sigma|} \{p^*_{\sigma(1)} f_1, ..., p^*_{\sigma(n)} f_n\} = \sum_{\sigma \in S_n} \{p^*_1 f_{\sigma(1)}, ..., p^*_n f_{\sigma(n)}\}$$

For $n \geq 2$ set

$$\beta_n : S^n k(E)^* \longrightarrow I^{2n}_{E(k)} \qquad f_1 \circ ... \circ f_n \longmapsto (f_1) * (f_2) * ... * (f_n)$$

Definition 5.2 *Let $k = \bar{k}$ and $n \geq 3$. Then $R_n(E/k)$ is the subgroup of $\mathbb{Z}[E]$ generated by $\beta_{n-1}(Ker \bar{\mu}_{n-1})$ and "distribution relations" :*

$$\{a\}_n - m^{n-2} \cdot \sum_{mb=a} \{b\}_n, \quad a \in E(k), \qquad m = -1, 2 \tag{32}$$

Example $R_3(E)$ is generated by the elements $(1-f)*(f)^-$, where $\{x\}^- := \{-x\}$, and (33).

Remark. It would be more natural to add to the subgroup $R_n(E/k)$ the distribution relations for all $m \in \mathbb{Z}, m \neq 0$. However we will get the same group, and for our purposes the definitioon above is technically more convinient.

Definition 5.3 *Let $k = \bar{k}$.*

$$B_n(E) := \frac{\mathbb{Z}[E(k)]}{R_n(E/k)}$$

Theorem 5.4 *$\delta(R_n(E)) \subset R_{n-1}(E) \otimes J$.*

The proof consists of two independent parts. For a more dificult one see proof of the theorem (5.9) below. The easy one follows from

Lemma 5.5 *For any $m \in \mathbb{Z}$, $m \neq 0$, one has*

$$\delta_n\Big(\{a\}_n - m^{n-2} \cdot \sum_{mb=a} \{b\}_n\Big) = 0 \quad \textit{in the group} \quad B_{n-1}(E) \otimes J \tag{33}$$

Proof. For $n = 2$ this is done in [GL]. The general case follows by induction:

$$\delta\Big(\{a\}_n - m^{n-2} \cdot \sum_{mb=a} \{b\}_n\Big) = \{a\}_{n-1} \otimes a - m^{n-3} \cdot \sum_{mb=a} \{b\}_{n-3} \otimes mb =$$

$$\Big(\{a\}_{n-1} - m^{n-3} \cdot \sum_{mb=a} \{b\}_{n-1}\Big) \otimes a = 0$$

So we get a homomorphism $\delta : B_n(E) \longrightarrow B_{n-1}(E) \otimes J$ and thus the following complex $B(S^{n-2}\mathcal{H}(1))^\bullet$:

$$B_n(E) \xrightarrow{\delta} B_{n-1}(E) \otimes J \xrightarrow{\delta} ... \xrightarrow{\delta} B_2(E) \otimes \Lambda^{n-2}J \xrightarrow{\delta} J \otimes \Lambda^{n-1}J \xrightarrow{\delta} \Lambda^n J \tag{34}$$

Here the very left group sits in degree one. The differential is defined by the formula (31) and has degree +1.

If k is not an algebraically closed we postulate the Galois descent:

$$B(E/k, n+2)^\bullet_{\mathbb{Q}} := (B(E/\bar{k}, n+2)^\bullet_{\mathbb{Q}})^{Gal(\bar{k}/k)}$$

Let us also define the groups $B^*_n(E)$ for $\bar{k} = k$:

$$B^*_n(E) := Im\Big(I^{2n-2} \longrightarrow B_n(E)\Big)$$

Here the map is induced by the natural inclusion $I^{2n-2} \hookrightarrow \mathbb{Z}[E]$.

It follows from the lemma (5.1) that $\delta_n(B^*_n(E)) \subset B^*_{n-1}(E) \otimes J$. So we can consider the following subcomplex $B^*(E, n)^\bullet$ of the complex (34).

$$B^*_n(E) \xrightarrow{\delta} B^*_{n-1}(E) \otimes J \xrightarrow{\delta} ... \xrightarrow{\delta} k^* \otimes \Lambda^{n-2}J$$

Proposition 5.6 *The canonical morphism of complexes*

$$B^*(E, n+2)^\bullet_{\mathbb{Q}} \longrightarrow B(E, n+2)^\bullet_{\mathbb{Q}}$$

is a quasiisomorphism.

Proof. This morphism is injective by the definition of the groups $B^*_n(E)$. It follows immediately from the lemma below that the quotient is isomorphic to the Koszul complex

$$S^n J_{\mathbb{Q}} \longrightarrow S^{n-1}J_{\mathbb{Q}} \otimes J_{\mathbb{Q}} \longrightarrow ... \longrightarrow S^2 J_{\mathbb{Q}} \otimes \Lambda^{n-2} J_{\mathbb{Q}} \longrightarrow J_{\mathbb{Q}} \otimes \Lambda^{n-1} J_{\mathbb{Q}} \longrightarrow \Lambda^n J_{\mathbb{Q}}$$

Lemma 5.7

$$B_n(E)/B^*_n(E) \otimes \mathbb{Q} = S^n J_{\mathbb{Q}}$$

Proof. We may assume that k is closed. We need to study the quotient of the group $\mathbb{Q}[E(k)]/I^{2n-2}$ along the subgroup generated by the distribution relations. Notice that

$$\mathbb{Q}[E(k)]/I^{2n-2} = \mathbb{Q} \oplus J_{\mathbb{Q}} \oplus S^2 J_{\mathbb{Q}} \oplus ... \oplus S^{2n-3} J_{\mathbb{Q}}$$

where the isomorphism is given by $i = (i_0, ..., i_{2n-3})$ where

$$i_m : \mathbb{Q}[E(k)] \longrightarrow S^m J_{\mathbb{Q}}; \qquad \{a\} \longmapsto a^m$$

Let us denote by DR_n the subgroup generated by the distribution relations (33). Then the homomorphism

$$i_{\hat{n}} = (i_0, ..., \hat{i}_n, ..., i_{2n-3}) : DR_n \longrightarrow \mathbb{Q} \oplus J_{\mathbb{Q}} \oplus ... \oplus \hat{S}^n J_{\mathbb{Q}} \oplus ... \oplus S^{2n-3} J_{\mathbb{Q}}$$

is surjective. Indeed,

$$i_k\Big(\{a\}_n - m^{n-2} \cdot \sum_{mb=a} \{b\}_n\Big) = (1 - m^{n-k}) \cdot a^k$$

In particular $i_n(DR_n) = 0$. The lemma and hence the proposition are proved.

3. The complex $B^*(E/k, n+1)^{\bullet}_{\mathbb{Q}}$ **is an** *elliptic deformation* **of the complex** $B(Spec(k), n)^{\bullet}_{\mathbb{Q}}$. Let me remind that the complex $B(Spec(k), n)^{\bullet}$ looks as follows:

$$\mathcal{B}_n(k) \longrightarrow \mathcal{B}_{n-1}(k) \otimes k^* \longrightarrow ... \longrightarrow \mathcal{B}_2(k) \otimes \Lambda^{n-2} k^* \longrightarrow \Lambda^n k^*$$

where $\mathcal{B}_n(k)$ is the quotient of $\mathbb{Z}[k^*]$ along a certain subgroup $\mathcal{R}_n$.

There is an important difference between these two complexes. The complex $B(Spec(k), n)^{\bullet}_{\mathbb{Q}}$ is defined directly in terms of a field k, while to define the complex $B(E/k, n+1)^{\bullet}_{\mathbb{Q}}$ we have to go to the algebraic closure of k and then postulate the Galois descent property. So in general they can only be quasiisomorphic.

Suppose $k = \bar{k}$. When E degenerates to $\Big(P^1, \{0\} \cup \{\infty\}\Big)$ the complex $B(E/k, n+1)^{\bullet}_{\mathbb{Q}}$ degenerates to a complex quasiisomorphic to $B(Spec(k), n)^{\bullet}_{\mathbb{Q}}$ (for $n = 2$ see s.3.4 in [GL]).

In a sence the elliptic situation is simpler. For example our definition of the group $R_n(E)$ is not inductive and more explicit then the definition of the group $\mathcal{R}_n(k)$ in [G2]. In fact when E degenerates to $\Big(P^1, \{0\} \cup \{\infty\}\Big)$ our definition suggests a new way of defining of the groups $\mathcal{B}_n(k)$ from [G1]-[G2].

4. Proof of the theorem (5.4). Let $\sigma \in S^n$ be a permutation and $a_i \in E$. Consider the following codimension p cycle in $E^{(n-1)}$:

$$X(\sigma; a_1, ..., a_p) := \{(x_1, ..., x_n) \subset E^{(n-1)} | x_{\sigma(1)} = a_1, ..., x_{\sigma(p)} = a_p\}$$

It is a product of elliptic curves E. Let us define a homomorphism

$$\mu_{n;p} : \Lambda^p \mathbb{Z}[E] \otimes S^{n-p} k(E)^* \longrightarrow \coprod_{X \in (E^{(n-1)})^p} \Lambda^{n-p}(k(X))^*$$

by setting

$$\{a_1\} \wedge ... \wedge \{a_p\} \otimes f_{p+1} \circ ... \circ f_n \longmapsto \sum_{\sigma \in S_n} (-1)^{|\sigma|} \{f_{p+1}(x_{\sigma(p+1)}), ..., f_n(x_{\sigma(n)})\}|_{X(\sigma; a_1, \ldots, a_p)}$$

Denote by $\tilde{\mathcal{D}}^p_{(n)}$ the image of the homomorphism $\mu_{n;p-1}$.

The elements of type $f_1(x_1) \wedge ... \wedge f_m(x_m)$ in $\Lambda^m k(E^{(m-1)})^*$ and their linear combinations may be called the *decomposable* elements; this suggests the notation.

Let F be an arbitrary field with a discrete valuation v and the residue class $\bar{F}_v$. The group of units U has a natural homomorphism $U \longrightarrow F_v^*$, $u \mapsto \bar{u}$. Choose a uniformizer $\pi \in F^*$, $\text{ord}_v \pi = 1$. There is canonical homomorphism

$$\Lambda^n(F^*) \xrightarrow{\partial_v} \Lambda^{n-1}(F_v^*)$$

uniquely defined by the properties

$$\partial_v(u_1 \wedge ... \wedge u_n) = 0; \qquad \partial_v(\pi \wedge u_2 \wedge ... \wedge u_n) = (\bar{u}_2, ..., \bar{u}_n)$$

Consider the following complex on $E^{(n-1)}$:

$$\Lambda^n(k(E^{(n-1)})) \xrightarrow{\partial} \coprod_{X \in (E^{(n-1)})^1} \Lambda^{n-1}(k(X)) \xrightarrow{\partial} ... \xrightarrow{\partial} \coprod_{X \in (E^{(n-1)})^{n-1}} k(X)^* \qquad (35)$$

Here $\partial := \sum_X \partial_{v(X)}$ where $v(X)$ is the valuation corresponding to the irreducible divisor X.

Lemma 5.8 *The groups $\tilde{\mathcal{D}}^p_{(n)}$ form a subcomplex in the complex (35)*

Proof. Clear from the definitions.

Let us define for $p < n$ a homomorphism

$$\tilde{\beta}_n^{(p+1)} : \tilde{\mathcal{D}}^{p+1}_{(n)} \longrightarrow I_E^{2(n-p+1)} \otimes \Lambda^p J$$

We define $\tilde{\beta}_n^{(p+1)}$ first on space of decomposable elements on the subvariety $x_1 = a_1, ..., x_p = a_p$ by the formula

$$\tilde{\beta}_n^{(p+1)} : f_{p+1}(x_{p+1}) \wedge ... \wedge f_n(x_n)|_{x_1=a_1,\ldots,x_p=a_p} \longmapsto$$

$$(f_{p+1}) * ... * (f_n) * (a_1 + ... + a_p) \otimes a_1 \wedge ... \wedge a_p$$

It extends uniquely to the space of all decomposable elements assuming the skewsymmetry with respect to the action S_n. In particulary it is defined on $\tilde{\mathcal{D}}^p_{(n)}$.

If $p = n-1$ then $\tilde{\mathcal{D}}^n_{(n)} = k^* \otimes \mathbb{Z}[E^{(n-1)}]^{sgn}$ and we have a homomorphism

$$\tilde{\beta}^{(n)}_n : k^* \otimes \mathbb{Z}[E^{(n-1)}]^{sgn} \longrightarrow k^* \otimes \Lambda^{n-1} J; \qquad x \otimes (a_1, ..., a_n) \longmapsto x \otimes a_1 \wedge ... \wedge a_{n-1}$$

Finally, one can define a homomorphism

$$I^4_E \otimes \Lambda^{n-2} J \stackrel{\delta}{\longrightarrow} k^* \otimes \Lambda^{n-1} J$$

Namely, there is a homomorphism

$$B_3(E) \otimes \Lambda^{n-2} J \stackrel{\delta}{\longrightarrow} B_{\mathcal{H}(1)}(E) \otimes \Lambda^{n-1} J \quad \{a\}_3 \otimes b_1 \wedge ... \longmapsto \{a\}_2 \otimes a \wedge b_1 \wedge ...$$

Let $B^{(k)}_n(E)$ be the subgroup of $B_n(E)$ generated by k-th degree of the augmentation ideal I^k_E. The restriction of this homomorphism to $B^{(4)}_3(E) \otimes \Lambda^{n-2} J \subset B_3(E) \otimes \Lambda^{n-2} J$ lands in $k^* \otimes \Lambda^{n-1} J$. Indeed, the composition

$$I^4_E \otimes \Lambda^{n-2} J \stackrel{\delta}{\longrightarrow} B_2(E) \otimes \Lambda^{n-1} J \longrightarrow S^2 J \otimes \Lambda^{n-1} J$$

is equal to zero.

Theorem 5.9 *The maps $\tilde{\beta}^{(i)}_n$ provide a homomorphism of complexes*

$$\begin{array}{ccccccccc}
\tilde{\mathcal{D}}^1_{(n)} & \stackrel{\partial}{\longrightarrow} & \tilde{\mathcal{D}}^2_{(n)} & \stackrel{\partial}{\longrightarrow} & ... & \stackrel{\partial}{\longrightarrow} & \tilde{\mathcal{D}}^n_{(n-1)} & \stackrel{\partial}{\longrightarrow} & \tilde{\mathcal{D}}^n_{(n)} \\
\downarrow \tilde{\beta}^{(1)}_n & & \downarrow \tilde{\beta}^{(2)}_n & & & & \downarrow \tilde{\beta}^{(n-1)}_n & & \downarrow \tilde{\beta}^{(n)}_n \\
I^{2n}_E & \stackrel{\delta}{\longrightarrow} & I^{2n-2}_E \otimes J & \stackrel{\delta}{\longrightarrow} & ... & \stackrel{\delta}{\longrightarrow} & I^4_E \otimes \Lambda^{n-2} J & \stackrel{\delta}{\longrightarrow} & k^* \otimes \Lambda^{n-1} J
\end{array}$$

Proof. We will do in details the commutativity of the left square, at the same time proving the theorem (5.4). The commutativity of the other squares except the very right one is completely similar.

The commutativity of the right square is a more subtle statement. For $n = 2$ it was already proved in [GL], s.3. The general case $n > 2$ is similar.

Consider an element

$$f_1(x_1) \wedge ... \wedge f_n(x_n) \in \Lambda^n k(E^{(n-1)}) \tag{36}$$

Let $v_a(f)$ be the order of zero of the function $f(t)$ at $t = a$.

The part of the coboundary (in the complex $\tilde{\mathcal{D}}^{\bullet}_{(n)}$) of the element (36) sitting on the divisor $x_1 = a_1$ is equal to

$$\sum_{a_1 \in E(k)} v_{a_1}(f_1) \cdot f_2(x_2) \wedge ... \wedge f_n(x_n), \quad x_2 + ... + x_n = -a_1 \tag{37}$$

Let t_a be the shift by a on the group $E(k)$, so $t_a\{b\} = \{a+b\}$ and $t_a f(x) = f(x-a)$ (sic). Then $(t_a f) = (f) * (a)$. Then it can be written as

$$\sum_{a_1 \in E(k)} v_{a_1}(f_1) \cdot f_2(y_2) \wedge ... \wedge t_{a_1} f_n(y_n), \quad y_2 + ... + y_n = 0$$

Applying the homomorphism $\tilde{\beta}_n^{(2)}$ to the element (37) we get

$$\sum_{a_1 \in E(k)} v_{a_1}(f_1) \cdot (f_2) * ... * (f_n) * (a_1) \otimes a_1 \tag{38}$$

From the other hand writing

$$(f_1) * ... * (f_n) = \sum_{a_i \in E(k)} v_{a_1}(f_1) \cdot ... \cdot v_{a_n}(f_n) \cdot \{a_1 + ... + a_n\}$$

we get

$$\delta((f_1) * ... * (f_n)) = \sum_{a_i \in E(k)} v_{a_1}(f_1) \cdot ... \cdot v_{a_n}(f_n) \cdot \{a_1 + ... + a_n\} \otimes (a_1 + ... + a_n) \tag{39}$$

Collecting the terms with a_1 in the right factor we get just what needed: the formula (38). Taking into consideration the coboundaries of the element (36) sitting on the divisors $x_p = a_p$ we will get the other terms (with a_p) in (39).

To prove the commutativity of the right square we replace it by a bigger diagram

$$\begin{array}{ccc} \tilde{\mathcal{D}}^n_{(n-1)} & \xrightarrow{\partial} & \tilde{\mathcal{D}}^n_{(n)} \\ \downarrow \tilde{\beta}_n^{(n-1)} & & \downarrow \tilde{\beta}_n^{(n)} \\ B_3(E) \otimes \Lambda^{n-2} J & \xrightarrow{\delta} & B_2(E) \otimes \Lambda^{n-1} J \end{array}$$

and then prove its commutativity in a way similar to the proof of theorem 4.5 in [GL]. The theorem is proved.

Consider the Gersten complex for the Milnor K-theory on $E^{(n-1)}$ ([NS]):

$$K_n^M(k(E^{(n-1)})) \xrightarrow{\partial} \coprod_{X \in (E^{(n-1)})^1} K_{n-1}^M(k(X)) \xrightarrow{\partial} ... \xrightarrow{\partial} \coprod_{X \in (E^{(n-1)})^{n-1}} k(X)^*$$

The Gersten complex is obtained from (35) by factorisation along the subgroup generated by the Steinberg relations. Let

$$\mathcal{D}^\bullet_{(n)} : \qquad \mathcal{D}^1_{(n)} \longrightarrow \mathcal{D}^2_{(n)} \longrightarrow ... \longrightarrow \mathcal{D}^n_{(n)}$$

be the image of the complex $\tilde{\mathcal{D}}^\bullet_{(n)}$ in the Gersten complex. In other words

$$\mathcal{D}^p_{(n)} = \tilde{\mathcal{D}}^p_{(n)} / St^p_{(n)}$$

where $St^p_{(n)}$ is the intersection of $\tilde{\mathcal{D}}^p_{(n)}$ with the subgroup generated by the Steinberg relations in $\coprod_{X\in (E^{(n-1)})^{p-1}} \Lambda^{n-p+1}k(X)^*$.

Lemma 5.10 $\beta_n^{(p)}(St^p_{(n)}) = 0$ *in* $B^*_{n+2-p}(E)\otimes\Lambda^{p-1}J$.

Proof. Consider the subvariety $X(id; a_1, ..., a_{p-1})$ in $E^{(n-1)}$ and its projection

$$p : X(id; a_1, ..., a_{p-1}) \longrightarrow E^{n-p} \qquad p : (x_1, ..., x_n) \longmapsto (x_p, ..., x_{n-1})$$

The subgroup of Steinberg relations is generated by

$$\{f(x_p, ..., x_{n-1}), (1-f)(x_p, ..., x_{n-1}), g_1(x_p, ..., x_{n-1}), ...\}$$

Notice that $x_n = -a_1 - ... - a_{p-1} - x_p - ... - x_{n-1}$. This means that an element $\sum_j \{f_p^{(j)}(x_p), ..., f_n^{(j)}(x_n)\}$ of the subgroup of Steinberg relations can be written as

$$p^*\sum_j \{f_p^{(j)}(y_p), ..., t_{-a_1-...-a_p} f_n^{(j)}(y_n)\}$$

on

$$E^{(n-p)} = \{(y_p, ..., y_n) | y_p + ... + y_n = 0\}$$

Now the lemma follows immediately.

Theorem 5.11 *There is a canonical homomorphism of complexes*

$$\begin{array}{ccccccc}
\mathcal{D}^1_{(n)} & \longrightarrow & \mathcal{D}^2_{(n)} & \longrightarrow & ... & \longrightarrow & \mathcal{D}^n_{(n)} \\
\downarrow \bar{\beta}_n^{(1)} & & \downarrow \bar{\beta}_n^{(2)} & & & & \downarrow \bar{\beta}_n^{(n)} \\
B^*_{n+1}(E) & \stackrel{\delta}{\longrightarrow} & B^*_n(E)\otimes J & \stackrel{\delta}{\longrightarrow} & ... & \stackrel{\delta}{\longrightarrow} & k^*\otimes\Lambda^{n-1}J
\end{array}$$

Proof. Follows immediately from lemma (5.10) and theorem (5.9).

6 The regulator integrals, Eisenstein - Kronecker series and a conjecture on $L(Sym^n E, n+1)$

1. Beilinson's conjecture in the case of $L(Sym^n h^1(E), n+1)$. Let E be an elliptic curve over a number field k. According to the general Beilinson conjecture on regulators the special value $L(Sym^n h^1(E), n+1)$ should be equal, up to standard factors, to the (co)volume of a certain lattice obtained as follows.

The n-th Deligne complex $\mathbb{R}(n)_{\mathcal{D}}$ on a regular variety X can be defined as the total complex associated with the following bicomplex:

$$\begin{array}{ccccccccc}
\mathcal{A}^0_X(n-1) & \stackrel{d}{\longrightarrow} & \dots & \stackrel{d}{\longrightarrow} & \mathcal{A}^n_X(n-1) & \stackrel{d}{\longrightarrow} & \mathcal{A}^{n+1}_X(n-1) & \stackrel{d}{\longrightarrow} & \dots \\
 & & & & \uparrow \pi_n & & \uparrow \pi_n & & \\
 & & & & \Omega^n_X & \stackrel{\partial}{\longrightarrow} & \Omega^{n+1}_X & \stackrel{\partial}{\longrightarrow} &
\end{array} \tag{40}$$

Here $(\mathcal{A}^\bullet_X, d)$ is the C^∞-De Rham complex, $\pi_n : \mathcal{A}^m_X \otimes \mathbb{C} \longrightarrow \mathcal{A}^m_X(n-1)$ is the projection induced by $\pi_n : \mathbb{C} \longrightarrow \mathbb{R}(n-1), \quad z \longmapsto z + (-1)^{(n-1)}\bar{z}$, the group $\mathcal{A}^0_X(n-1)$ is placed in degree 1 and $(\Omega^\bullet_X, \partial)$ is the De Rham complex of holomorphic forms with logarithmic singularities at infinity.

One has the regulator map

$$r_{Be} : H^n_{\mathcal{M}}\Big(E^{(n-1)}, \mathbb{Q}(n)\Big)_{sgn} \longrightarrow H^n_{\mathcal{D}}\Big(E^{(n-1)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(n)\Big)_{sgn} \tag{41}$$

The right hand side is a group of purely topological origin:

$$H^n_{\mathcal{D}}\Big(E^{(n-1)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(n)\Big)_{sgn} = H^{n-1}_B\Big(E^{(n-1)} \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(n-1)\Big)^+_{sgn} =$$

$$\Big(\oplus_{\sigma: k \hookrightarrow \mathbb{C}} H^{n-1}_B(E^{(n-1)}_\sigma(\mathbb{C}), \mathbb{R}(n-1))_{sgn}\Big)^+ =$$

$$\Big(\oplus_{\sigma: k \hookrightarrow \mathbb{C}} Sym^{n-1} H^1_B(E_\sigma(\mathbb{C}), \mathbb{R}(1))\Big)^+$$

Here $+$ means invariants of the complex conjugation acting on σ's and on the coefficients $\mathbb{R}(n-1)$.

The image of the regulator map (41) is conjectured to be a lattice. $H^n_{\mathcal{D}}\Big(E^{(n-1)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{Q}(n)\Big)_{sgn}$ gives another lattice in $H^n_{\mathcal{D}}\Big(E^{(n-1)} \otimes_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(n)\Big)_{sgn}$. The covolume of the lattice $Imre_{Be}$ measured with respect to the second lattice should coincide (up to standard factors) with the special value of our L-function at $s = n+1$.

2. The Eisenstein-Kronecker series. Let me recall their definition:

$$K_{i,j}(z;\tau) = \sum_{\gamma \in \Gamma \setminus 0} \frac{(z,\gamma)}{\gamma^i \bar{\gamma}^j}, \quad i,j \geq 1$$

For the relation with the elliptic polylogarithms see [BL], [Z].

Lemma 6.1 *a) For any lattice* Γ *one has* $\bar{K}_{i,j}(z;\tau) = (-1)^{i+j} K_{j,i}(z;\tau)$

b) Suppose that the lattice Γ *and a divisor* P *on* $\mathbb{C}/\Gamma$ *are invariant under the complex conjugation. Then* $K_{i,j}(P;\tau) \in \mathbb{R}$.

Proof. Clear.

Consider for each $n > 2$ a homomorphism

$$K_n : \mathbb{Z}[E(\mathbb{C})] \longrightarrow Sym^{n-2} H^1(E(\mathbb{C}), \mathbb{C})$$
$$\{z\} \longmapsto \sum_{a+b=n} K_{a,b}(z;\tau)(dz)^{a-1}(d\bar{z})^{b-1}$$

Theorem 6.2 $K_n(R_n(E(\mathbb{C}))) = 0$.

We will prove it in s 6.3 below. So we get a homomorphism

$$K_n : B_n[E(\mathbb{C})] \longrightarrow Sym^{n-2} H^1(E(\mathbb{C}), \mathbb{C})$$

This means that $R_n(E(\mathbb{C}))$ is a subgroup of functional equations for the Eisenstein-Kronecker series.

Lemma 6.3 *Suppose that* E *is defined over* $\mathbb{R}$ *and the lattice* Γ *was defined using a real differential* ω. *Then*

$$K_n : B_n[E(\mathbb{C})]^+ \longrightarrow Sym^{n-2} H^1(E(\mathbb{C}), \mathbb{R}(1))^+$$

Here $+$ means invariants of the complex involution acting on both $\mathbb{R}(1)$ and $E(\mathbb{C})$.

Proof. Follows from lemma (6.1).

3. Computation of the regulator integral. The main result of this section is due to Deninger (see [D1], s.6). Our presentation is technically simpler since working with distributions we avoid the convergence problems.

For any functions $f_1, ..., f_n$ on a manifold X consider the following $(n-1)$-form with values in $\mathbb{R}(n-1) := (2\pi i)^{n-1}\mathbb{R}$

$$r_n(f_1, ..., f_n) := \tag{42}$$

$$\text{Alt}_n \sum_{j \geq 0} C_j \cdot \log|f_1| d\log|f_2| \wedge ... \wedge d\log|f_{2j+1}| \wedge di\arg f_{2j+2} \wedge ... \wedge di\arg f_n$$

Here $C_j := \frac{1}{(2j+1)!(n-2j-1)!}$ and Alt_n is the operation of alternation of f_i's. Then

$$dr_n(f_1, ..., f_n) = \pi_n(d\log f_1 \wedge ... \wedge d\log f_n)$$

This just means that the pair

$$(r_n(f_1, ..., f_n), \quad d\log f_1 \wedge ... \wedge d\log f_n)$$

is an n-cycle in the Deligne complex (40). It is the product in the real Deligne cohomology of the 1-cocycles $(\log|f_i|, d\log f_i)$. Set

$$\omega_{p,q} := \Big(\sum \overset{(-)}{dz_1} \wedge ... \wedge \overset{(-)}{dz_{p+q}}\Big)^{(p,q)} \tag{43}$$

where $\overset{(-)}{dz_i}$ means either dz_i or $\bar{d}z_i$ and the sum is taken over all possible terms. For example

$$\omega_{2,0} = dz_1 \wedge dz_2, \quad \omega_{1,1} = dz_1 \wedge d\bar{z}_2 + d\bar{z}_1 \wedge dz_2, \quad \omega_{0,2} = d\bar{z}_1 \wedge d\bar{z}_2$$

The forms $\pi_n \omega_{p,q}$ for $p \geq q, p+q = n-1$, form a basis over $\mathbb{R}$ in $H_B^{n-1}(E_{/\mathbb{R}}^{(n-1)}, \mathbb{R}(n-1))_{sgn}$.

We can represent elements in $H_B^{n-1}(E_{/\mathbb{R}}^{(n-1)}, \mathbb{R}(n-1))_{sgn}$ by their cup product with forms $\pi_n \omega_{p,q}$:

$$\frac{1}{(2\pi i)^{(n-1)}} \int_{E^{(n-1)}(\mathbb{C})} \sum_{\sigma \in S_n} (-1)^{|\sigma|} r_n(p_{\sigma(1)}^* f_1, ..., p_{\sigma(n)}^* f_n) \wedge \pi_n \omega_{p,q}$$

Theorem 6.4

$$\int_{E^{(n-1)}(\mathbb{C})} \sum_{\sigma \in S_n} (-1)^{|\sigma|} r_n(p_{\sigma(1)}^* f_1, ..., p_{\sigma(n)}^* f_n) \wedge \omega_{p,q} = c_n \frac{Im\tau}{\pi} \cdot K_{p+1,q+1}(f_1 * ... * f_n)$$

where $c_n \in \mathbb{Q}^*$.

The constant c_n can be obtained from the proof below.

Proof. It consists of several reductions of the integral.

Step 1. The form $\omega_{p,q}$ is skew invarant under the action of the group S_n. So the integral is equal to

$$n! \cdot \int_{E^{(n-1)}(\mathbb{C})} r_n(p_1^* f_1, ..., p_n^* f_n) \wedge \omega_{p,q}$$

Step 2. Let

$$\alpha_n(f_1, ..., f_n) := \sum_{i=1}^{n} (-1)^i \log|f_i| d\log|f_1| \wedge ... d\widehat{\log}|f_i| ... \wedge d\log|f_n| \tag{44}$$

Lemma 6.5 *For any functions* $f_1, ..., f_n$

$$\int_{E^{(n-1)}(\mathbb{C})} r_n(f_1, ..., f_n) \wedge \omega_{p,q} = b_n \cdot \int_{E^{(n-1)}(\mathbb{C})} \alpha_n(f_1, ..., f_n) \wedge \omega_{p,q}$$

where $b_n \in \mathbb{Q}^*$ *is a (computable) constant.*

Proof. One always have either $d\log f_i \wedge \omega_{p,q} = 0$ or $d\log \bar{f}_i \wedge \omega_{p,q} = 0$. So we can replace everywhere $di \arg f$ by $\pm d\log|f|$.

Step 3.

$$\int_{E^{(n-1)}(\mathbb{C})} \alpha_n(f_1, ..., f_n) \wedge \omega_{p,q} = n \cdot \int_{E^{(n-1)}(\mathbb{C})} \log|f_n| d\log|f_1| \wedge ... \wedge d\log|f_{n-1}| \wedge \omega_{p,q} \tag{45}$$

Indeed, $\log|f_1|d\log|f_2| + \log|f_2|d\log|f_1| = d(\log|f_1|\cdot\log|f_2|)$ and so by the Stokes theorem

$$\int_{E^{(n-1)}(\mathbb{C})} d(\log|f_1|\cdot\log|f_2|)d\log|f_3|\wedge ...\wedge d\log|f_n|\wedge\omega_{p,q} = 0$$

Step 4. Let us compute the right integral in (45).

Lemma 6.6

$$\log|f(z)| = -\frac{Im\tau}{2\pi}\sum_{\gamma\in\Gamma\backslash 0} v_a(f)\frac{(z-a,\gamma)}{|\gamma|^2} + C_f \tag{46}$$

where C_f is a certain constant.

Proof. One can get a proof applying $\partial\bar{\partial}$ to the both parts of (45). The constant C_f can be computed from the decomposition of f on the product of theta functions using the formula in s. 18 ch. VIII of [We]. According to step 5 it does not play any role in our considerations.

Step 5. By the Stokes formula

$$\int_{E^{(n-1)}(\mathbb{C})} C_f\cdot d\log|f_1|\wedge ...\wedge d\log|f_{n-1}|\wedge\omega_{p,q} = 0$$

we see that one can neglect the constants C_f.

Step 6. We may suppose that in (45) the form $\omega_{p,q}$ is written in variables $z_1,...,z_{n-1}$. Then for each summand in $\omega_{p,q}$ one can replace $d\log|f_i|$ by $1/2\cdot\partial\log f_i$ or $1/2\cdot\bar{\partial}\log f_i$ depending whether in this summand appeared $d\bar{z}_i$ or dz_i.

For example for the summand $dz_1\wedge...\wedge dz_p\wedge d\bar{z}_{p+1}\wedge...\wedge d\bar{z}_{n-1}$ we will have the integral

$$\frac{n}{2^{n-1}}\cdot\int_{E^{(n-1)}(\mathbb{C})}\log|f_n(z_n)|\bar{\partial}\log f_1(z_1)\wedge ...\wedge\bar{\partial}\log f_p(z_p)\wedge$$

$$\partial\log f_{p+1}(z_{p+1})\wedge ...\wedge\partial\log f_{n-1}(z_{n-1})\wedge dz_1\wedge ...\wedge dz_p\wedge d\bar{z}_{p+1}\wedge ...\wedge d\bar{z}_{n-1}$$

Differentiating the distributions we get

$$\partial\log f(\bar{z}) = \sum_{\gamma\in\Gamma\backslash 0} v_a(f)\frac{(z-a,\gamma)\cdot\bar{\gamma}}{|\gamma|^2}$$

$$\bar{\partial}\log f(z) = -\sum_{\gamma\in\Gamma\backslash 0} v_a(f)\frac{(z-a,\gamma)\cdot\gamma}{|\gamma|^2}$$

The condition $z_1 + ... + z_n = 0$ just mean that we compute the value of the convolution of one variable distributions:

$$\log|f_n(z)| * \frac{\bar{\partial}\log f_2(\bar{z})}{\partial\bar{z}} * ... * \frac{\bar{\partial}\log f_p(\bar{z})}{\partial\bar{z}} * \frac{\partial\log f_p(z)}{\partial z} * ... * \frac{\partial\log f_{n-1}(z)}{\partial z}$$

at 0. Using the fact that the Fourier transform sends the convolution to the product and the formulas above we get the theorem.

Recall that we have a homomorphism $K_{n+1} : \mathbb{Z}[E(\mathbb{C})] \longrightarrow Sym^{n-1}H^1_B(E(\mathbb{C}), \mathbb{C})$ which is constructed as follows

$$\{z\} \longmapsto \sum_{i=0}^{n-1} K_{a,b}(z) \cdot (dz)^{a-1}(d\bar{z})^{b-1} \in Sym^{n-1}H^1_B(E(\mathbb{C}), \mathbb{C})$$

Theorem (6.2) claims that it sends the subgroup $R_{n+1}(E(\mathbb{C}))$ to zero.

4. Proof of the theorem (6.2). For any function f on a manifold one has $d\log|f| \wedge d\log|1-f|$. So a non zero term in the integral (6.5) could be

$$\int_{E^{(n-1)}(\mathbb{C})} (\log|f|d\log|1-f| - \log|1-f|d\log|f|) \wedge d\log|f_3| \wedge ... \wedge d\log|f_n| \wedge \omega_{p,q}$$

One always has

$$(\log|f|d\log|1-f| - \log|1-f|d\log|f|) \wedge \omega_{p,q} =$$

$$\pm i \cdot (\log|f|darg(1-f) - \log|1-f|dargf) \wedge \omega_{p,q}$$

Further one has, even in the sence of the distribution,

$$d\mathcal{L}_2(f) = \log|f|darg(1-f) - \log|1-f|dargf$$

So we can rewrite the integral as

$$\pm i \cdot \int_{E^{(n-1)}(\mathbb{C})} d(\mathcal{L}_2(f) \wedge d\log|f_3| \wedge ... \wedge d\log|f_n| \wedge \omega_{p,q}) = 0$$

It is zero by the Stokes formula.

Now using theorem (6.4) relating Eisenstein-Kronecker series to the regulator integral we come to the proof of the theorem.

Theorem (6.2) and lemma (6.1) imply

Theorem 6.7 *The Eisenstein-Kronecker map K_{n+1} provides a homomorphism*

$$K_{n+1} : B_{n+1}[E(\bar{\mathbb{Q}})]^{Gal(\bar{\mathbb{Q}}/F)} \longrightarrow \Big(\oplus_{\sigma: F \hookrightarrow \mathbb{C}} Sym^{n-1}H^1_B(E_\sigma(\mathbb{C}), \mathbb{R}(1))\Big)^+$$

Combining these results with conjecture (1.4), our construction of the elliptic motivic complexes presented in s. 4.1 and Beilinson's conjecture on regulators we come to conjecture (6.8) about L-function of $Sym^{n-1}h^1(E)$ at $s = n$ for an arbitrary elliptic curve E over a number field.

5. A conjecture on $L(Sym^n E, n+1)$ In this section we will assume for simplicity that E be an elliptic curve over $\mathbb{Q}$. We will left to reader as an easy exercise to generalise all the discussion to the case of an elliptic curve over an arbitrary field F.

Conjecture (1.4) together with Beilinson's conjecture on regulators imply a precise conjecture on $L(Sym^n E, n+1)$

For any divisor $P = \sum n_s(P_s)$ on $E(\mathbb{C})$ put $K_{i,j}(P) := \sum n_s K_{i,j}(P_s)$.

The integrality condition . Suppose E has a split multiplicative reduction at p with N-gon as a special fibre. Let L be a finite extention of $\mathbb{Q}_p$ of degree $n = ef$ and $\mathcal{O}_L$ the ring of integers in L. Let E^0 be the connected component of the Néron model of E over $\mathcal{O}_L$. Let us fix an isomorphism $E^0_{F_{p^f}} = \mathbb{G}_m/F_{p^f}$. It provides a bijection between $\mathbb{Z}/eN\mathbb{Z}$ and the components of $E_{F_{p^f}}$. For a divisor P such that all its points are defined over L denote by $d(P;\nu)$ the degree of the restriction of the flat extension of a divisor P to the ν'th component of the (eN)-gon.

Let $B_{n+1}(x)$ be the $(n+1)$-th Bernoulli polynomial. The integrality condition at p is the following condition on a divisor P, provided by the work of Schappaher and Scholl ([SS]). For a certain (and hence for any, see s. 3.3 in [GL]) extention L of $\mathbb{Q}_p$ such that all points of the divisor P are defined over L one has ($[L:\mathbb{Q}_p] = ef$):

$$\sum_{\nu \in \mathbb{Z}/(eN)\mathbb{Z}} d(P;\nu) B_{n+1}(\frac{\nu}{eN}) = 0 \tag{47}$$

Let C_n be the conductor of the system of the l-adic representations related to $Sym^n h^1(E)$. Set

$$\beta_{2l+1} = C_{2l+1}^{-(l+1)} \cdot (Im\tau)^{(l+1)(l+2)}, \qquad \beta_{2l} = C_{2l}^{-\frac{2l+1}{2}} \cdot \pi^{-2l}(Im\tau)^{(l+1)^2}$$

Conjecture 6.8 *a) For any elliptic curve over $\mathbb{Q}$ there exist $[\frac{n}{2}]+1$ $\mathbb{Q}$-rational divisors P_a on $E(\bar{\mathbb{Q}})$ such that*

$$L(Sym^n h^1(E), n+1) \sim_{\mathbb{Q}^*} \beta_n det|K_{b,n+2-b}(P_a;\tau)| \tag{48}$$

($1 \leq a,b \leq [\frac{n}{2}]+1$), and the divisors P_a satisfy the following two conditions:

i) $$\delta(P_a) = 0 \quad in \quad B_{n+1}(E) \otimes J(\bar{\mathbb{Q}})_{\mathbb{Q}} \tag{49}$$

ii) the integrality condition*: at each prime p where E has a split multiplicative reduction with special fibre a Neron N-gon*

$$\sum_{\nu \in \mathbb{Z}/(eN)\mathbb{Z}} d(P;\nu) B_{n+1}(\frac{\nu}{eN}) = 0 \tag{50}$$

b) For any $[\frac{n}{2}]+1$ $\mathbb{Q}$-rational divisors P_a on $E(\bar{\mathbb{Q}})$, satisfying the conditions above the right hand side of (48) is equal to $q \cdot L(S^n h^1(E), n+1)$ where q is a rational number, perhaps equal to 0.

In [W] J.Wildeshaus, assuming standard conjectures about mixed motives, formulated a conjecture similar to the part b) of the conjecture (6.8) (an elliptic analog of the weak version of Zagier's conjecture).

For $n = 2$ the formula (48) was proved for modular elliptic curves over $\mathbb{Q}$ in [GL]. Formula (48), even without precise conditions on the divisors P_a, is the most nontrivial part of the conjecture for $n > 2$ (see also s.8 in [G4]). An efficient way to formulate the conditions on the divisors P_a without referring to the definition of the subgroups $R_n(E)$ is given in the chapter 7.

When E degenerates to the nodal curve, the conjecture on $L(Sym^2 E, 3)$ leads to Zagier's conjecture [Za2] at $s = 3$, which was proved in [G1]-[G2]. This gives a credit for the conjecture (6.8).

The key condition (49) is obviously satisfied if P_a are (multiples of) torsion divisors. The determinants from (48) for torsion divisors where considered by Deninger ([De2], s.5) (and inspired by the Eisenstein symbol of Beilinson [Be]). They work well for CM elliptic curves. However Mestre and Schappacher [SM] deduced from a result of Serre that for a given non CM elliptic curve over $\mathbb{Q}$ for all $n > n_0$ the determinant is always zero for any $\mathbb{Q}$-rational torsion divisors P_a. So to get the L-values one has to consider the non torsion divisors.

6. A more explicit form of the conditions on the divisors P_a. Let $P = \sum n_i P_i$ and $k(P)$ be the field generated by the points P_i. Let us denote by h_v the canonical local height related to a valuation v of a number field K. Let K_v be the completion of a number field K corresponding to v. Recall the height homomorphism $h_v : B_2(E(K_v)) \longrightarrow \mathbb{R}$. If v is a non archimedean valuation then the target of this homomorphism is $(\log p) \cdot \mathbb{Q}$.

Let us consider a homomorphism

$$d_m : B_{n+1}(E) \longrightarrow B_m(E) \otimes S^{n+1-m} J_{\mathbb{Q}}, \qquad \{a\}_{n+1} \longmapsto \{a\}_{n+1-m} \otimes a \cdot ... \cdot a$$

We will need the following pairs of homomorphisms. If $m = 2$:

i) A homomorphism

$$p_2 \otimes id : B_2(E) \otimes S^{n-1} J_{\mathbb{Q}} \longrightarrow S^2 J_{\mathbb{Q}} \otimes S^{n-1} J_{\mathbb{Q}}$$

where $p_2 : \{a\}_2 \longmapsto a \cdot a$.

For any valuation v of we have

ii) The height homomorphism

$$h_v \otimes id : B_2(E) \otimes S^{n-1} J_{\mathbb{Q}} \longrightarrow \mathbb{R} \otimes_{\mathbb{Q}} S^{n-1} J_{\mathbb{Q}}$$

where v is any valuation of the field $K(P)$.

For $m > 2$:

iii) The Bernoulli homomorphism, defined for any bad prime where E has a split multiplicative reduction with the Néron N-gon:

$$Ber_m : B_m(E) \otimes S^{n+1-m} J_{\mathbb{Q}} \longrightarrow S^{n+1-m} J_{\mathbb{Q}},$$

$$\{a\}_m \otimes b_1 \cdot ... \cdot b_{n+1-m} \longmapsto \sum_{\nu} d(a,\nu) B_m(\frac{\nu}{N}) \cdot b_1 \cdot ... \cdot b_{n+1-m}$$

iv) The Eisenstein-Kronecker homomorphism

$$K_m \otimes id : B_m(E) \otimes S^{n+1-m} J_{\mathbb{Q}} \longrightarrow Sym^{n-2} H^1(E(\mathbb{C}), \mathbb{R}(1))^+ \otimes S^{n+1-m} J_{\mathbb{Q}}$$

Remarks. 1. In formulas above $J_{\mathbb{Q}}$ means $J(\bar{\mathbb{Q}})_{\mathbb{Q}}$. However for a given divisor P we land in $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant part of powers of $J(k(P))_{\mathbb{Q}}$.

2. Let us consider the Eisenstein-Kronecker homomorphism only on the kernel of the Bernoulli homomorphism. Then Beilinson's conjecture on regulators means that it should land in

$$\Big(\text{the regulator lattice in} \quad Sym^{n-2} H^1(E(\mathbb{C}), \mathbb{R}(1))^+\Big) \otimes S^{n+1-m} J_{\mathbb{Q}}$$

So by the Mordell-Weil theorem if $k(P)$ is a given number field then the target group is a finite dimensional $\mathbb{Q}$-vector space.

3. If v is a p-adic valuation of the field $k(P)$, then $(\log p)^{-1} \cdot h_v(P_i) \in \mathbb{Q}$, and so the target of the height homomorphism is a finite dimensional $\mathbb{Q}$-vector space.

4. If the a divisor P is in the kernel of the height homomorphisms for all archimedean valuations but one then thanks to the product formula it is sent to zero by all of them. In particulary if $k(P) = \mathbb{Q}$ we can forget the archimedian valuation.

Composing each of these homomorphism with the appropriate map d_m we get homomorphisms

$$(p_2 \otimes id) \circ d_2, \quad (h_v \otimes id) \circ d_2, \quad Ber_m \circ d_m, \quad (K_m \otimes id) \circ d_m \qquad (51)$$

here $m > 2$.

Proposition 6.9 *The condition $\delta(P) = 0$ in the group $B_n(E) \otimes J_{\mathbb{Q}}$ implies that all of the homomorphisms (51) are equal to zero.*

Proof. Clear.

The height condition is the crucial one. If it is satisfied, then for a given field $k(P_a)$ the other conditions should give only a finite number of conditions on the divisors P_a.

If a $\mathbb{Q}$-rational divisor P is sent to zero by all of the homomorphisms (51) then this essentially means that $\delta(P) = 0$ in the group $B_n(E) \otimes J(\bar{\mathbb{Q}})_{\mathbb{Q}}$. To see this we write the homomorphism d_m as a composition of homomorphisms $\delta \otimes id$

$$B_{n+1}(E) \longrightarrow B_n(E) \otimes J_{\mathbb{Q}} \longrightarrow (B_{n-1}(E) \otimes J_{\mathbb{Q}}) \otimes J_{\mathbb{Q}} \longrightarrow ... \longrightarrow B_m(E) \otimes J_{\mathbb{Q}}^{\otimes n+1-m} \qquad (52)$$

followed by the projection

$$B_m(E) \otimes J_{\mathbb{Q}}^{\otimes n+1-m} \longrightarrow B_m(E) \otimes S^{n+1-m} J_{\mathbb{Q}}$$

Let us spell the details in the first interesting case: $n = 3$.

Proposition 6.10 *Let us assume that the Bloch-Beilinson regulator $r_{Be} : K_2(E)_{\mathbb{Z}} \longrightarrow \mathbb{R}$ is injective. Then if for $n = 3$ a $\mathbb{Q}$-rational divisor P belongs to the kernel of homomorphisms (51) then $\delta(P) = 0$ in the group $B_3(E) \otimes J_{\mathbb{Q}}$*

Proof. Consider the homomorphism

$$d_2 : B_4(E) \longrightarrow B_2(E) \otimes S^2 J, \qquad \{a\}_4 \longmapsto \{a\}_2 \otimes a \cdot a \tag{53}$$

Suppose an element $P \in B_4(E)$ is in the kernel of the homomorphism $(p_2 \otimes id) \circ d_2$. Then $P \in B_4^*(E)$ and $f(P) \in (\bar{\mathbb{Q}}^* \otimes S^2 J(\bar{\mathbb{Q}}))^{Gal(\bar{\mathbb{Q}}/\mathbb{Q}}$.

The image of the divisor P under this homomorphism belongs to the subgroup $k(P)^* \otimes S^2 J(k(P))$. The hight condition is a way to say that it is equal to zero.

Let us write the map d_2 as a composition

$$B_4(E) \longrightarrow B_3(E) \otimes J \longrightarrow (B_2(E) \otimes J) \otimes J \longrightarrow B_2(E) \otimes S^2 J$$

Notice that

$$Ker\Big(B_3(E)\otimes J \longrightarrow (B_2(E)\otimes J)\otimes J\Big)\otimes\mathbb{Q} = Ker\Big(B_3(E) \longrightarrow B_2(E)\otimes J\Big)\otimes J\otimes\mathbb{Q}$$

Consider the homomorphism

$$B_3(E(\mathbb{C})) \otimes J(\mathbb{C}) \longrightarrow J(\mathbb{C}) \otimes \mathbb{R}, \qquad b \otimes \{a\}_3 \longmapsto K_{2,1}(a) \otimes b \tag{54}$$

There is a homomorphism $K_2(E(\mathbb{C})) \longrightarrow B_3(E(\mathbb{C}))$ such that the following diagram is commutative (see [GL]):

$$\begin{array}{ccc} K_2(E(\mathbb{C})) & \xrightarrow{r_{Be}} & \mathbb{R} \\ \downarrow & & \downarrow Id \\ B_3(E(\mathbb{C})) & \xrightarrow{K_3} & \mathbb{R} \end{array}$$

So assuming the injectivity of the regulator $K_2(E)_{\mathbb{Z}} \longrightarrow \mathbb{R}$ we see that (54) should be injective on $Ker\Big(B_3(E) \longrightarrow B_2(E) \otimes J\Big) \otimes J \otimes \mathbb{Q}$.

7 The complexes $\mathcal{B}(E;n)^\bullet$ and motivic elliptic polylogarithms

1. In this chapter $k = \bar{k}$, and all abelian groups are tensored by $\mathbb{Q}$, so we work with the corresponding $\mathbb{Q}$-vector spaces. For instence $J := J \otimes \mathbb{Q}$ etc.

Let

$$\mathbb{Q}[E(k)] \xrightarrow{\delta} \mathbb{Q}[E(k)] \otimes J, \qquad \{a\} \longmapsto \{a\} \otimes a$$

Theorem 7.1 *Let us assume standard conjectures on mixed motives. Then there exist canonical homomorphisms* $l_n : \mathbb{Q}[E(k)] \to \mathcal{L}(E)_{S^{n-2}\mathcal{H}(1)^\vee}$ *such that the following digram is commutative:*

$$\begin{array}{ccc} \mathbb{Q}[E(k)] & \stackrel{\delta}{\longrightarrow} & \mathbb{Q}[E(k)] \otimes J \\ l_{n-1} \downarrow & & \downarrow l_{n-2} \otimes id \\ \mathcal{L}(E)_{S^{n-2}\mathcal{H}(1)^\vee} & \stackrel{\partial}{\longrightarrow} & \mathcal{L}(E)_{S^{n-3}\mathcal{H}(1)^\vee} \otimes J \end{array}$$

Proof. The proof is based on the existence and basic properties of the motivic elliptic polylogarithms of Beilinson and Levin [BL]. For any nonzero point $a \in E(k)$ let $G_a^{(1)}$ be an element of $Ext^1_{\mathcal{M}_E}(\mathbb{Q}(0), \mathcal{H})$ wich corresponds to $a \in J$ under the isomorphism $Ext^1_{\mathcal{M}_E}(\mathbb{Q}(0), \mathcal{H}) = J$. Set $G_a^{(m)} := S^m(G_a^{(1)})$. The motivic elliptic $(n-1)$-logarithm at a is a mixed elliptic motive $El_{n-1}(a)$ which provides a certain extension class in $Ext^1_{\mathcal{M}_E}(\mathcal{H}, G_a^{(n-1)}(1))$. In particular its weight graded quotients are

$$\mathcal{H}, \mathbb{Q}(1), \mathcal{H}(1), ..., S^{(n-1)}\mathcal{H}(1)$$

Therefore it has canonical framing and so defines an element of $\mathcal{A}(\mathcal{H}, S^{(n-1)}\mathcal{H}(1))$. After tensoring it by $\mathcal{H}$ and twisting by $\mathbb{Q}(-1)$ we can introduce a natural framing by $\mathbb{Q}$ and $S^{(n-2)}\mathcal{H}(1)$ (since $S^{(n-1)}\mathcal{H} \otimes \mathcal{H} = S^{(n)}\mathcal{H} \oplus S^{(n-2)}\mathcal{H}$). Therefore we picked up an element $l_{n-1}(a) \in \mathcal{A}(\mathbb{Q}, S^{(n-2)}\mathcal{H}(1))$.

The commutativity of the diagram follows from the properties of the elliptic polylogarithms ([BL]). The crucial point is this. Since $W_{\leq -3}El_{n-1}(a)$ is a symmetric power of the motive $G_a^{(1)}$, The projection to $\mathcal{L}(E)$ of the elements in $\mathcal{A}(S^{(k)}\mathcal{H}(1), S^{(n-1)}\mathcal{H}(1))$ coming from the canonical framing by $S^{(k)}\mathcal{H}(1)$ and $S^{(n-1)}\mathcal{H}(1))$ of the motive $El_{n-1}(a)$ are zero provided $0 \leq k < n-2$. So projecting the coproduct of $El_{n-1}(a)$ to $\Lambda^2\mathcal{L}(E)$ the only nonzero contribution we get is given by the component of the coproduct coming from $\mathcal{A}(\mathcal{H}, S^{(n-2)}\mathcal{H}(1)) \otimes \mathcal{A}(S^{(n-2)}\mathcal{H}(1), S^{(n-1)}\mathcal{H}(1))$. The fact that it is equal to $l_{n-2}(a) \wedge a$ follows immediately from the basic property of the elliptic polylogarithm motive (see [BL]).

Definition 7.2

$$\mathcal{R}_n(E) = Kerl_{n-1}, \qquad \mathcal{B}_n(E) = \frac{\mathbb{Q}[E(k)]}{\mathcal{R}_n(E)}$$

Theorem (7.1) implies that δ provides a well defined homomorphism $\delta : \mathcal{B}_n(E) \longrightarrow \mathcal{B}_{n-1}(E) \otimes J$. So we get a complex $\mathcal{B}(E;n)^\bullet$:

$$\mathcal{B}_n(E) \longrightarrow \mathcal{B}_{n-1}(E) \otimes J \longrightarrow ... \longrightarrow \mathcal{B}_2(E) \otimes \Lambda^{n-2}J \longrightarrow J \otimes \Lambda^{n-1}J \longrightarrow \Lambda^n J$$

Set

$$r_n(J) := Ker\Big(J \otimes \Lambda^{n-1} J \longrightarrow \Lambda^n J\Big)$$

The theorem (7.1) and this definition immediately imply that there exists canonical homomorphism of complexes

$$\begin{array}{ccccccc}
\mathcal{B}_n(E) & \longrightarrow & \mathcal{B}_{n-1}(E) \otimes J & \longrightarrow ... \longrightarrow & \mathcal{B}_2(E) \otimes \Lambda^{n-2} J & \longrightarrow r_n(J) \\
l_{n-1} \downarrow & & l_{n-2} \otimes id \downarrow & & \downarrow l_1 \otimes id & || \\
\mathcal{L}(E)_{S^{n-2}\mathcal{H}(1)^\vee} & \longrightarrow & \mathcal{L}(E)_{S^{n-3}\mathcal{H}(1)^\vee} \otimes J & \longrightarrow ... \longrightarrow & \mathcal{L}(E)_{\mathbb{Q}(1)^\vee} \otimes \Lambda^{n-2} J & \longrightarrow r_n(J)
\end{array}$$

Lemma 7.3 *The bottom complex is a subcomplex of the $S^{n-2}\mathcal{H}(1)^\vee$-isotypical component of the standard cochain complex of the Lie coalgebra $\mathcal{L}(E)$.*

So if $k = \bar{k}$, we get a canonical injective morphism of the complexes

$$\mathcal{B}(E; n+1)^\bullet \longrightarrow \Big(\Lambda^\bullet \mathcal{L}(E), \partial\Big)_{S^{n-1}\mathcal{H}(1)^\vee} \tag{55}$$

Let $\mathcal{K}^M_{n-1}$ be the sheaf of Milnor K-groups. Combining this morphism with the canonical morphism from the right hand side to $RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^{n-1}\mathcal{H}(1))$ provided by the inclusion functor $\mathcal{M}_E \to \mathcal{MM}_k$ we get

Corollary 7.4 *Let us assume the formalism of mixed motives. Then*

a) there exists canonical homomorphisms

$$H^i(\mathcal{B}(E; n+1)^\bullet_{\mathbb{Q}}) \longrightarrow gr^\gamma_n K_{n+1-i}(E^{(n-1)})_{sgn} \otimes \mathbb{Q} \tag{56}$$

b) The homomorphism for $i = 1$ is injective.

Indeed, thanks to (55) this is true if $k = \bar{k}$. The general case follows since we have the descent property both for rational K-theory and, (by definition), for the complexes $\mathcal{B}(E; n+1)^\bullet_{\mathbb{Q}}$.

Remark. We do not expect a morphism of complexes (55) exist if k is not algebraicly closed. The reason is this. If k is not closed we have postulated the Galois descent for the complexes $\mathcal{B}(E; n+1)^\bullet$. On the other hand the standard complex of the Lie algebra $\mathcal{L}(E/k)$ should not satisfy the Galois decent.

I hope a stronger result should be valid:

Conjecture 7.5 *a)*

$$gr^\gamma_n K_{n+1-i}(E^{(n-1)})_{sgn} \otimes \mathbb{Q} = H^{i-1}(E^{(n-1)}, \mathcal{K}^M_{n-1})_{sgn} \otimes \mathbb{Q} \tag{57}$$

b) There exists a canonical isomorpism in the derived category

$$\mathcal{B}(E; n+1)^\bullet_{\mathbb{Q}} = RHom_{\mathcal{MM}_k}(\mathbb{Q}(0), S^n\mathcal{H}(1))$$

in particular

$$H^i\Big(\mathcal{B}(E,n+1)^\bullet_{\mathbb{Q}}\Big) = gr^\gamma_n K_{n+1-i}(E^{(n-1)})_{sgn} \otimes \mathbb{Q} \tag{58}$$

c) Let $k=\bar{k}$. Then the homomorphism of complexes (55) is a quasiisomorphism.

If the conjecture (4.2) is correct, then b) is equivalent to c).
According to the lemma (7.3) this conjecture implies conjecture (1.4).
Part a) of the conjecture is trivial for $n=2$.

Lemma 7.6 *For $n=3$ and $k=\bar{k}$ the part a) of the conjecture (7.5) follows from the rigidity conjecture for $K_3^{ind}(k)$.*

Proof. The statement boils down to $Ker\beta = Im\alpha$ in the sgn-part of the diagram

$$\begin{array}{cccc}
\longrightarrow & B_2(k(E^{(2)}))\otimes k(E^{(2)})^* & \longrightarrow & \Lambda^3 k(E^{(2)})^* \\
& \downarrow \alpha & & \downarrow \\
& \coprod_{Y\in(E^{(2)})^1} B_2(k(Y)) & \stackrel{\beta}{\longrightarrow} & \coprod_{Y\in(E^{(2)})^1} \Lambda^2 k(Y)^* \\
& \downarrow & & \downarrow \\
& 0 & & ...
\end{array}$$

Let $\beta = \coprod_Y \beta_Y$. Then $Ker\beta_Y = K_3^{ind}(k(Y))_{\mathbb{Q}}$. By the rigidity conjecture any point $y\in Y$ provides an isomorphism $K_3^{ind}(k)_{\mathbb{Q}} = K_3^{ind}(k(Y))_{\mathbb{Q}}$. So $Ker\beta/Im\alpha$ is a subgroup of

$$Coker\Big(K_3^{ind}(k)_{\mathbb{Q}}\otimes k(E^{(2)})^* \longrightarrow \coprod_{Y\in(E^{(2)})^1} K_3^{ind}(k)_{\mathbb{Q}}\Big) = CH^1(E^{(2)})_{sgn}\otimes\mathbb{Q} = 0$$

The lemma is proved.

2. Motivic realization of elliptic polylogarithms. Let $f=(f_1,...,f_n)$ be an n - tuple of rational functions on E.

Motivation. Consider the following multivalued analytic function on

$$\{n-\text{tuples of rational functions on } \quad E(\mathbb{C})\}\times H_{n-1}(E^{(n-1)}(\mathbb{C}))_{sgn_n}$$

$$P(E^{(n-1)};f;\gamma) := \int_\gamma \mathrm{Alt}_{(x_1,\dots,x_n)} \log f_1(x_1))d\log f_2(x_2)\wedge...\wedge d\log f_n(x_n) \tag{59}$$

where γ is a cycle representing a nontrivial class in $H_{n-1}(E^{(n-1)}(\mathbb{C}))_{sgn_n}$. The subscript sgn_n means the skewsymmetric part with respect to the permutations. (A better way to define this function is given by formula (64) below.) We will show that this function is a period of a mixed elliptic motive.

Choose a coordinate z on $\mathbb{P}^1$. Let Δ_n be the coordinate cube in $(\mathbb{P}^1; 0 \cup \infty)^n$, i.e. union of $2n$ hyperplanes $z_i = 0$, $z_i = \infty$. Notice that $(\mathbb{P}^1)^n \backslash \Delta_n = (\mathbb{G}_m)^n$.

Let us define a codimension n cycle

$$Z(E^{(n-1)}; f) \subset E^{(n-1)} \times (\mathbb{P}^1)^n$$

as follows:

$$Z(E^{(n-1)}; f) := \mathrm{Alt}_{(x_1,...,x_n)}\{x_1, ..., x_n; f_1(x_1), ..., f_n(x_n)\} \cup E^{(n-1)} \times \{1, ..., 1\}$$

Here we use the coordinate system $z_1, ..., z_n$, i.e. $z_i = f_i(x_i)$. The group S_n acts on $(\mathbb{G}_m)^n$ by permutations. So we get an action of the group $S_n \times S_n$ on $E^{(n-1)} \times (\mathbb{G}_m)^n$. In this chapter we mark by the subscript sgn the skewsymmetric part under this action. Consider the following mixed motive

$$h(E^{(n-1)}; f) := H^n(E^{(n-1)} \times (\mathbb{G}_m)^n, Z^0(E^{(n-1)}; f))(n)_{sgn}$$

where $Z^0(E^{(n-1)}; f) := Z(E^{(n-1)}; f) \backslash \Big(Z(E^{(n-1)}; f) \cap \Delta_n \Big)$.

More precisely, $h(E^{(n-1)}; f)(-n) := R^n p_! \mathcal{F}(E; f)_{sgn}$ where $\mathcal{F}(E; f)$ is the following mixed motivic sheaf on $E^{(n-1)} \times (\mathbb{P}^1)^n$. Take the constant sheaf on the complement to $Z(E^{(n-1)}; f) \cup E^{(n-1)} \times \Delta_n$ in $E^{(n-1)} \times (\mathbb{P}^1)^n$; extend it by j_* to the divisor $E^{(n-1)} \times \Delta_n$ and then by $j_!$ to $Z(E^{(n-1)}; f)$.

Lemma 7.7 *$h(E^{(n-1)}; f)$ is a mixed elliptic motive framed by $\mathbb{Q}(0)$ and $Sym^{n-1}\mathcal{H}(1)$.*

Proof. i) $\mathbb{Q}(0)$*-component of the frame.* Let us prove that

$$gr^W_{2n} H^n(E^{(n-1)} \times (\mathbb{G}_m)^n, Z^0(E^{(n-1)}; f))_{sgn} = \mathbb{Q}(-n)$$

Notice that $H^n(\mathbb{G}_m)^n_{sgn} = H^n(\mathbb{G}_m)^n$ and $H^i(\mathbb{G}_m)^n_{sgn} = 0$ for $i < n$. So the projection $E^{(n-1)} \times (\mathbb{G}_m)^n \longrightarrow (\mathbb{G}_m)^n$ induces an isomorphism

$$\mathbb{Q}(-n) = gr^W_{2n} H^n((\mathbb{G}_m)^n)_{sgn} \stackrel{\sim}{\longrightarrow} gr^W_{2n} H^n(E^{(n-1)} \times (\mathbb{G}_m)^n)_{sgn}$$

The canonical morphism

$$H^n(E^{(n-1)} \times (\mathbb{G}_m)^n, Z^0(E^{(n-1)}; f))_{sgn} \longrightarrow H^n(E^{(n-1)} \times (\mathbb{G}_m)^n)_{sgn}$$

induces an isomorphism after taking gr^W_{2n}. Indeed, there is an exact sequence

$$H^{n-1} Z^0(E^{(n-1)}; f) \longrightarrow H^n(E^{(n-1)} \times (\mathbb{G}_m)^n, Z^0(E^{(n-1)}; f))_{sgn} \longrightarrow$$

$$H^n(E^{(n-1)} \times (\mathbb{G}_m)^n)_{sgn} \longrightarrow H^n Z^0(E^{(n-1)}; f)$$

and

$$gr^W_{2n} H^{n-1} Z^0(E^{(n-1)}; f) = gr^W_{2n-1} H^{n-1} Z^0(E^{(n-1)}; f) = 0$$

since $Z^0(E^{(n-1)}; f)$ is an open regular variety of dimension $n-1$.

ii) $Sym^{n-1}\mathcal{H}(1)$*-component of the frame.* One has

$$H^n(E^{(n-1)} \times (\mathbb{P}^1)^n, Z(E^{(n-1)}; f))_{sgn} = Sym^{n-1}h^1(E) \tag{60}$$

Indeed, $H^n(E^{(n-1)} \times (\mathbb{P}^1)^n)_{sgn} = 0$. So there is an exact sequence

$$H^{n-1}(E^{(n-1)} \times (\mathbb{P}^1)^n)_{sgn} \xrightarrow{\alpha} H^{n-1}(Z(E^{(n-1)}; f))_{sgn} \xrightarrow{\beta}$$
$$H^n(E^{(n-1)} \times (\mathbb{P}^1)^n, Z(E^{(n-1)}; f))_{sgn} \longrightarrow 0$$

Further,

$$H^{n-1}(E^{(n-1)} \times (\mathbb{P}^1)^n)_{sgn} = Sym^{n-1}h^1(E)$$
$$H^{n-1}(Z(E^{(n-1)}; f))_{sgn} = Sym^{n-1}h^1(E) \oplus Sym^{n-1}h^1(E)$$

and α is injective.

The restriction map induces an isomorphism

$$gr^W_{n-1}H^n(E^{(n-1)} \times (\mathbb{P}^1)^n, Z(E^{(n-1)}; f))_{sgn} \longrightarrow \tag{61}$$
$$gr^W_{n-1}H^n(E^{(n-1)} \times (\mathbb{G}_m)^n, Z^0(E^{(n-1)}; f))_{sgn}$$

because $(Z(E^{(n-1)}; f)$ is regular of dimension $n-1$).

$$W_{n-1}H^{n-1}Z^0(E^{(n-1)}; f) = W_{n-1}H^{n-1}Z(E^{(n-1)}; f)$$

Combining (60) and (61) we get the $Sym^{n-1}\mathcal{H}(1)$-component of the frame. The lemma is proved.

Finally, we show that $h(E^{(n-1)}; f)$ is a mixed *elliptic* motive by induction using the following basic observation: intersection of the cycle $Z(E^{(n-1)}; f_1, ..., f_n)$ with any codimension 1 face of the cube Δ_n is a sum of cycles of form $Z(E^{(n-2)}; g_1, ..., g_{n-1})$. For example

$$Z(E^{(n-1)}; f_1, ..., f_n) \cap \{z_1 = 0\} = \sum_{x \in E} m_x(f_1) \cdot Z(E^{(n-2)}; f_2, ..., f_n)$$

where $m_x(f)$ is the multiplicity of zero at x. The lemma is proved.

Remark 7.8 *One can apply the same construction to n arbitrary functions $f_1(z_1, ..., z_n), ..., f_n(z_1, ..., z_n)$ on $E^{(n-1)}$. However it is not clear whether the motive $h(E^{(n-1)}; f_1, ..., f_n)$ is a mixed elliptic motive in general.*

The functions $f_1, ..., f_n$ on the E define a map

$$f : E^{(n-1)} \longrightarrow (\mathbb{P}^1)^n, \quad (x_1, ..., x_n) \longmapsto (f_1(x_1), ..., f_n(x_1))$$

Let $E_f^{(n-1)}$ be the image of this map and $\tilde{E}_f^{(n-1)} := E_f^{(n-1)} \backslash \Big(E_f^{(n-1)} \cap \Delta\Big)$. Consider the following motive:

$$\tilde{h}(E^{(n-1)}; f) := H^n((\mathbb{G}_m)^n, \tilde{E}_f^{(n-1)})(n)_{sgn} \tag{62}$$

Lemma 7.9 *Suppose that* $f_*: H_{n-1}(E^{(n-1)})_{sgn} \to H_{n-1}(E_f^{(n-1)})_{sgn}$ *is a nonzero map. Then*

a) $\tilde{h}(E^{(n-1)}; f)$ *is a* $(\mathbb{Q}(0), Sym^{n-1}\mathcal{H}(1))$*-framed mixed elliptic motive.*

b) $\tilde{h}(E^{(n-1)}; f) = h(E^{(n-1)}; f)$ *as framed motives.*

c) If $f_* = 0$ *then* $h(E^{(n-1)}; f) = 0$.

Proof. a) is very similar to the proof of the lemma above. For instance the $Sym^{n-1}h^1(E)$ part of the framing comes from isomorphism

$$W_{n-1}H^n((\mathbb{P}^1)^n \backslash \Delta_n, \tilde{E}_f^{(n-1)}) \longrightarrow Sym^{n-1}h^1(E)$$

Namely, $H^i(P^n)_{sgn} = 0$, so there is an isomorphism

$$H^{n-1}(E_f^{(n-1)})_{sgn} \longrightarrow H^n((\mathbb{P}^1)^n, E_f^{(n-1)})_{sgn}$$

Combining it with $f^*: H^{n-1}(E_f^{(n-1)})_{sgn} \longrightarrow H^{n-1}(E^{(n-1)})_{sgn}$ we get a morphism

$$H^n((\mathbb{P}^1)^n, E_f^{(n-1)})_{sgn} \longrightarrow H^{n-1}(E^{(n-1)})_{sgn} = Sym^{n-1}h^1(E) \qquad (63)$$

b) The projection

$$E^{(n-1)} \times (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n$$

induces a morphism respecting the frames.

c) is clear from the construction. The lemma is proved.

The period corresponding to this framing is exactly the function (59). Indeed, consider the differential form

$$\omega_{\Delta_n} := d\log(z_1) \wedge ... \wedge d\log(z_n)$$

in $(\mathbb{CP}^1)^n \backslash \Delta_n$. Let Γ be a relative n-chain in $\mathbb{C}P^n$ which bounds an $(n-1)$-cycle γ, $[\gamma] \in H_{n-1}((\mathbb{CP}^1)^n, E_f^{(n-1)})_{sgn}$. Then

$$L_n(E^{(n-1)}; f; \gamma) = \int_\Gamma \omega_{\Delta_n} \qquad (64)$$

The Stokes formula shows that the integrals (64) and (59) coinside.

Proposition 7.10 *The* $\mathbb{R}$*-valued period of the Hodge realization of* $h(E^{(n-1)}; f)$ *is given by*

$$\mathcal{L}(E^{(n-1)}; f) := \int_{E^{(n-1)}(\mathbb{C})} \mathrm{Alt}_{x_1,...,x_n} r_n(f_1, ..., f_n) \wedge \omega_{p,q}$$

This integral coincides with the one computed by means of the Eisenstein - Kronecker series, as was explained before.

Recall that $\mathcal{A}(\mathbb{Q}, Sym^{n-1}\mathcal{H}(1))$ is the group of mixed elliptic motives framed by $\mathbb{Q}(0)$ and $Sym^{n-1}\mathcal{H}(1)$.

Lemma 7.11 *For any $\lambda_i \in k^*$ one has the equality of framed motives $h(E^{(n-1)}; f_1, ..., f_n) = h(E^{(n-1)}; \lambda_1 \cdot f_1, ..., \lambda_n \cdot f_n)$*

Proof. The action of an element $g = (\lambda_1, ..., \lambda_n) \in \mathbb{G}_m^n$ on $P^n \backslash \Delta_n = \mathbb{G}_m^n$ provides a morphism of motives $h(E^{(n-1)}; \lambda_1 \cdot f_1, ..., \lambda_n \cdot f_n) \longrightarrow h(E^{(n-1)}; f_1, ..., f_n)$ which obviously preserves the framing.

Theorem 7.12 *There is a well defined homomorphism of abelian groups*

$$m_n^* : S^n k(E)^* \longrightarrow \mathcal{A}(\mathbb{Q}(0), Sym^{n-1}\mathcal{H}(1)) \qquad f_1 \circ ... \circ f_n \longmapsto h(E^{(n-1)}; f)$$

It is zero if one of the functions f_i is a constant; one has $m_n^*(Ker\mu_n) = 0$.

Proof.

Below a generalization of the construction above is given. Let D^0 be the group of degree zero divisors on E. For any $d := (d_1, ..., d_n)$ let us construct a mixed elliptic motive $h(E^{(n-1)}; d) := h(E^{(n-1)}; d_1, ..., d_n)$ framed by $\mathbb{Q}(0)$ and $Sym^{n-1}\mathcal{H}(1)$.

Let $\mathcal{P}$ be the rigidified Poincaré line bundle over $J \times J$. For any two degree zero divisors d_1, d_2 with disjoint support there is an element

$$< d_1, d_2 > \in \mathcal{P}_{[d_1],[d_2]}$$

where $[d_i] \in J$ is the class of a degree zero divisor d_i.

Consider the following $(n-1)$-cycle

$$Z(E^{(n-1)}; d) \subset E^{(n-1)} \times \mathcal{P}^n \qquad (65)$$

$$c(E^{(n-1)}; d) := \text{Alt}_{(x_1, ..., x_n)}(x_1, ..., x_n; < d_1, (x_1) - (0) >; ...; < d_n, (x_n) - (0) >)$$

Here $x_1 + ... + x_n = 0$ and $(x_1, ..., x_n)$ belongs to Zariski open part of $E^{(n-1)}$ where the divisors d_i and $(x_i) - (0)$ are disjoint. and set

$$h(E^{(n-1)}; d) := H^n\Big(E^{(n-1)} \times \mathcal{P}^n, Z(E^{(n-1)}; d)\Big)(n)$$

Theorem 7.13 *a) $h(E^{(n-1)}; d)$ is a mixed elliptic motive framed by $\mathbb{Q}(0)$ and $Sym^{n-1}\mathcal{H}(1)$.*

b) There is a well defined homomorphism of abelian groups

$$S^n D^0 \longrightarrow \mathcal{A}(\mathbb{Q}(0), Sym^{n-1}\mathcal{H}(1)) \qquad d_1 \circ ... \circ d_n \longmapsto h(E^{(n-1)}; d)$$

Proof. Restriction to a fiber of the Poincare line bundle provides an isomorphism

$$gr_2^W H^1(\mathcal{P}) \longrightarrow H^1(\mathbb{G}_m) = \mathbb{Q}(-1)$$

and thus we get a first part of the framing:

$$\mathbb{Q}(-n) \longrightarrow gr_{2n}^W H^n(\mathcal{P}^n) \xrightarrow{\pi^*} gr_{2n}^W H^n(E^{(n-1)} \times \mathcal{P}^n)$$

where $\pi : E^{(n-1)} \times \mathcal{P}^n \to \mathcal{P}^n$ is the natural projection.

The second part of the framing comes from the fundamental cycle of $E^{(n-1)}$ just as before. The rest is straitforward.

3. Motivic construction of the complex $B^*(E;3)$. Recall the convolution map $\beta_2 : S^2k(E)^* \longrightarrow B_3^*(E), f_1 \circ f_2 \longmapsto (f_1) * (f_2)$. We are going to show that the diagram

$$\begin{array}{ccc} S^2k(E)^* & & \\ \downarrow \beta_2 & \searrow^{m_2^*} & \\ B_3^*(E) & \stackrel{l_2^*}{\longrightarrow} & \mathcal{L}^*(E)_{\mathcal{H}(1)} \end{array}$$

provides a well defined homomorphism

$$l_2^* : B_3^*(E) \longrightarrow \mathcal{L}^*(E)_{\mathcal{H}(1)}$$

(Here $\mathcal{L}^*(E)_{\mathcal{H}(1)^\vee} = \mathcal{L}^*(E)_{\mathcal{H}(1)} \boxtimes \mathcal{H}(1)^\vee$, so $\mathcal{L}^*(E)_{\mathcal{H}(1)}$ is a $\mathbb{Q}$-vector space.)

Consider the map

$$\mu_2 : S^2k(E)^* \longrightarrow \frac{K_2(K(E)_-}{\{k^*, k(E)^*\}_-}, \quad f_1 \circ f_2 \longmapsto \{f_1(x), f_2(-x)\} - \{f_1(-x), f_2(x)\}$$

According to theorem 3.9 in [GL] one has

Theorem 7.14 $\mu_2(Ker\beta_2) = 0$

It remains to use that $l_2^*(f * (1-f)) = 0$ by theorem 7.12.

Theorem 7.15 *We get a commutative diagram*

$$\begin{array}{ccc} B_3^*(E)_{\mathbb{Q}} & \stackrel{\delta^*_{\mathcal{H}(1)}}{\longrightarrow} & (k^* \otimes J)_{\mathbb{Q}} \\ l_2^* \downarrow & & \downarrow = \\ \mathcal{L}^*(E)_{\mathcal{H}(1)} & \stackrel{\delta}{\longrightarrow} & \mathcal{L}^*(E)_{\mathbb{Q}(1)} \otimes \mathcal{L}^*(E)_{\mathcal{H}} \end{array}$$

To prove this theorem we need only to compute $\delta h(E; f_1, f_2)$, which is left to the reader.

8 Elliptic Chow polylogarithms and generalized Eisenstein - Kronecker series

1. Elliptic Chow polylogarithms. *The single valued version.* Let C be a codimension n cycle in $E^k \times (\mathbb{P}^1)^l$, $k+l = 2n-1$, skewinvariant under the action of $G_k \times G_l$.

Recall the forms $\pi_n \omega_{p,q}$ (see (43)), which for $p \geq q, p+q = n-1$ form a basis over $\mathbb{R}$ in $H_B^{n-1}(E_{/\mathbb{R}}^{n-1}, \mathbb{R}(n-1))_{sgn_n}$. We represent elements in $H_B^{n-1}(E_{/\mathbb{R}}^{n-1}, \mathbb{R}(n-1))_{sgn_n}$ by their cup product with the forms $\pi_n \omega_{p,q}$:

The single valued elliptic Chow polylogarithm is a function

$$\mathcal{P}_{k,l} : \mathcal{Z}^n(E^k, l) \longrightarrow H_B^{n-1}(E_{/\mathbb{R}}^{n-1}, \mathbb{R}(n-1))_{sgn_n} \qquad k+l = 2n-1$$

defined as follows. Let $\pi : C \to (\mathbb{P}^1)^l$ and $p : C \to E^k$. If $k > 0$:

$$< \mathcal{P}_{k,l}(C), \omega_{p,q} > := \int_C \pi^* r_{k-1}(z_1, ..., z_k) \wedge p^* \omega_{p,q} \quad p \geq q, p+q = n-1,$$

Here we integrate over the nonsingular part of the complex points of the cycle C. The integral is always convergent (see [G6]). For example

$$\mathcal{P}_{2,1}(C) := \mathrm{Alt}_{(x_2,x_3)} \int_C \pi_1^* \log|z_1| \pi_2^* \omega \wedge \pi_3^* \bar{\omega}$$

The multivalued elliptic Chow polylogarithm, denoted $P_{k,l}(C)$, is defined as follows. Let $(x_1, ..., x_k, z_{k+1}, ..., z_{k+l})$ be the coordinates on $E^k \times (\mathbb{P}^1)^l$. Assume $l \neq 0$. Let π_i (resp. p_j) be the projection of C to the i-th coordinate $\mathbb{P}^1$ (resp. j-th factor E) in $E^k \times (\mathbb{P}^1)^l$.

i). *Assume* $k \leq n$. Then

$$P_{k,l}(C) :=$$

$$\mathrm{Alt}_{(G_k \times G_l)} \int_{p_1^*\gamma \times ... \times p_k^*\gamma \times \pi_{k+1}\delta \times ... \times \pi_{k+l}\delta} \log z_{n+1} d\log z_{n+2} \wedge ... \wedge d\log z_{2n-1}$$

ii). *Assume* $n < k < 2n-1$. Then

$$P_{k,l}(C) :=$$

$$\mathrm{Alt}_{(G_k \times G_l)} \int_{p_1^*\gamma \times ... \times p_n^*\gamma} p_{n+1}^*\omega \wedge ... \wedge p_k^*\omega \wedge (\log z_{k+1} d\log z_{k+2} \wedge ... \wedge d\log z_{2n-1})$$

Example 1. The multivalued elliptic Chow dilogarithms:

$$P_{0,3}(C) := \mathrm{Alt}_{(G_3)} \int_{\pi_1^*\delta} \log z_2 d\log z_3$$

$$P_{1,2}(C) := \mathrm{Alt}_{(G_1 \times G_2)} \int_{p_1^*\gamma} \log z_2 d\log z_3$$

$$P_{2,1}(C) := \mathrm{Alt}_{(G_2 \times G_1)} \int_{p_1^*\gamma} p_2^*\omega \cdot \log z_3$$

Example 2. The multivalued elliptic Chow trilogarithms:

$$P_{0,5}(C) := \mathrm{Alt}_{(G_5)} \int_{\pi_1^*\delta\times\pi_2^*\delta} \log z_3 d\log z_4 \wedge d\log z_5$$

$$P_{1,4}(C) := \mathrm{Alt}_{(G_1\times G_4)} \int_{p_1^*\gamma\times\pi_2^*\delta} \log z_3 d\log z_4 \wedge d\log z_5$$

$$P_{2,3}(C) := \mathrm{Alt}_{(G_2\times G_3)} \int_{p_1^*\gamma\times p_2^*\gamma} \log z_3 d\log z_4 \wedge d\log z_5$$

$$P_{3,2}(C) := \mathrm{Alt}_{(G_3\times G_2)} \int_{p_1^*\gamma\times p_2^*\gamma} p_3^*\omega \wedge \log z_4 \wedge d\log z_5$$

$$P_{4,1}(C) := \mathrm{Alt}_{(G_4\times G_1)} \int_{p_1^*\gamma\times p_2^*\gamma} p_3^*\omega \wedge p_4^*\omega \cdot \log z_5$$

The multivalued elliptic Chow polylogarithms are periods of mixed motives, which are easy to write down.

2. Some interesting cycles. Let $\mathbb{L}_n(a)$ be the codimension n cycle in $(\mathbb{P}^1)^{2n-1}$ responsible for the classical n-logarithm (see [Bl6] and [BK]):

$$\mathbb{L}_n(a) := \{x_1, ..., x_{k-1}, 1-x_1, 1-x_2/x_1, ..., 1-x_{k-1}/x_{k-2}, 1-a/x_{k-1}\} \in \mathcal{Z}^n(2n-1)$$

Consider the following cycle in $\mathcal{Z}^n(E^{(n-k-1)}, k+n)$:

$$\mathrm{Alt}_{(x_1,...,x_{n-k})}\Big(x_1, ..., x_{n-k}, \mathbb{L}_k(f_1(x_1)), f_2(x_2), ..., f_{n-k}(x_{n-k}\Big) \qquad (66)$$

Notice that we are using this $E^{(n-k-1)}$, not E^{n-k-1}.

Examples of cycles.

$$\text{in} \quad \mathcal{Z}^2(E,2): \quad \mathrm{Alt}_{(x_1,x_2)}\{x_1, f_1(x_1), f_2(x_2)\}$$

$$\text{in} \quad \mathcal{Z}^2(3): \quad \{z_1, 1-z_1, 1-a/z_1\}$$

$$\text{in} \quad \mathcal{Z}^3(E^{(2)},3): \quad \mathrm{Alt}_{(x_1,x_2,x_3)}\{x_1, x_2, f_1(x_1), f_2(x_2), f(x_3)\}$$

$$\text{in} \quad \mathcal{Z}^3(E,4): \quad \mathrm{Alt}_{(x_1,x_2)}\{x_1, 1-x_1, 1-f_1(x_1)/x_1, f_2(x_2)\}$$

$$\text{in} \quad \mathcal{Z}^3(5): \quad \{z_1, z_2, 1-z_1, 1-z_2/z_1, 1-a/z_2\}$$

I think the single valued elliptic Chow polylogarithm on these cycles should provide the new transcendental functions needed for $L(Sym^{n-k-1}E, n)$.

3. The generalized Eisenstein-Kronecker series and $L(Sym^nE, n+m)$ **for** $m \geq 1$. Conjecture 2.1 in [G2] for the field $k(E^{(n)})$ tells us that $H^{n+1}_{\mathcal{M}}(Speck(E^{(n)}), \mathbb{Q}(n+m))$ is generated by the sums of the elements of form

$$\sum_i \{f_0^{(i)}(x)\}_m \otimes f_1^{(i)}(x) \wedge ... \wedge f_n^{(i)}(x) \quad \text{in} \quad \mathcal{B}_m(k(E^{(n)})) \otimes \Lambda^n k(E^{(n)})^*$$

satisfying the condition

$$\sum_i \{f_0^{(i)}(x)\}_{m-1} \wedge f_0^{(i)}(x) \wedge f_1^{(i)}(x) \wedge ... \wedge f_n^{(i)}(x) \quad \text{in} \quad \mathcal{B}_{m-1}(k(E^{(n)})) \otimes \Lambda^{n+1} k(E^{(n)})^* \tag{67}$$

Definition 8.1 *$\mathcal{D}^{0,1}_{(n,m)}$ is the subgroup of $\mathcal{B}_m(k(E^{(n)})) \otimes \Lambda^n k(E^{(n)})^*$ generated by the elements*

$$\text{Alt}_{(x_0,...,x_n)}\{f(x_0)\}_m \otimes g_1(x_1) \wedge ... \wedge g_n(x_n) \tag{68}$$

where $f, g_k \in k(E)^*$, $x_0 + ... + x_n = 0$.

Denote by X the element (68). Set

$$\partial(X) := \text{Alt}_{(x_0,...,x_n)}\{f_0(x_0)\}_{m-1} \wedge f_0(x_0) \wedge g_1(x_1) \wedge ... \wedge g_n(x_n)$$

It belongs to the group $\mathcal{B}_{m-1}(\mathbb{C}(E^{(n)})) \otimes \Lambda^{n+1}\mathbb{C}(E^{(n)})^*$.

$$r(X) := \sum_{i=1}^{n}(-1)^i \sum_{a \in E(k)} v_a g_i \cdot \text{Alt}_{(x_0,...,x_n)}\{f_0(x_0)\}_m \otimes g_1(x_1) \wedge ... \hat{g}_i ... \wedge g_n(x_n)_{|x_i=a}$$

Here $v_a g$ is the valuation of g at the point a. Let $d_{n,m} := \partial + r$. $r(X)$ lies in the sum of the groups $\mathcal{B}_m(\mathbb{C}(X_{i,a}))^* \otimes \Lambda^{n-1}\mathbb{C}(X_{i,a})^*$ where $X_{i,a}$ is the divisor $x_i = a$.

Conjecture 8.2 *a) There exists a map*

$$\text{Ker} d_{n,m} \longrightarrow H^{n+1}_{\mathcal{M}}(E^{(n)}, \mathbb{Q}(n+m))_{sgn}$$

which in the case $k = \mathbb{C}$ commutes with the regulator map.
b) One might hope that this map is surjective.

Part a) of this conjecture can be deduced from standard conjectures. If $m = 1$ this is exactly conjecture discussed in chapter 5. If $n = 1$ the conjecture follows from the conjecture 2.1 in [G2], see also conjecture 8 in [G4] and $n = 1, m = 2$ it is proved in [G3]. In general the main argument for the hope expressed in the part b) is simplicity of the ansatz used to define the elements (68).

Element (68) provides a cycle of type (66), as was explaned above. The value of the regulator on the element which lies in $\text{Ker} d_{n,m}$ should coinside with the value of the elliptic Chow polylogarithm $\mathcal{P}_{n,m}$ on the corresponding cycle. Thus to get the generalized Eisenstein-Kronecker series responsible for the special values $L(Sym^n E, n+m)$ we evaluate the Chow polylogarithm $\mathcal{P}_{n,m}$ on the cycle (66) using the Fourier decomposition of $\mathcal{L}_k(f(x))$ and then the same method as in chapter 6. This boils down to computation of the following regulator integral $(p+q=n)$

$$r(\{f\}_m \otimes g_1 \wedge ... \wedge g_n; \bar{\omega}_{p,q}) := \int_{E^n(\mathbb{C})} \log|f|^{m-2} \cdot \alpha_2(1-f,f) \wedge \alpha_n(g_1,...,g_n) \wedge \omega_{p,q}$$

Set

$$r(\{f\}_m \otimes g_1 \wedge ... \wedge g_n) :=$$

$$\sum_{p+q=n} r(\{f\}_m \otimes g_1 \wedge ... \wedge g_n; \omega_{q,p})(dz)^p (d\bar{z})^q \in Sym^n H^1(E(\mathbb{C}), \mathbb{C})$$

Let

$$\Big(\mathbb{Z}[E] \otimes \mathbb{Z}[E] \otimes \mathbb{Z}[E]\Big)_E = \Big(\mathbb{Z}[E \times E \times E]\Big)_E$$

be the abelian group generated by the elements $\{x,y,z\}$ where $x,y,z \in E(k)$, subject to the relations $\{x,y,z\} = \{x+c, y+c, z+c\}$ for any $c \in E(k)$. Define

$$\beta_{n,m} : \{f\}_m \otimes g_1 \circ ... \circ g_n \longmapsto \Big(\mathbb{Z}[E] \otimes \mathbb{Z}[E] \otimes \mathbb{Z}[E]\Big)_E$$

$$\{f\}_m \otimes g_1 \circ ... \circ g_n \longmapsto (1-f) \otimes (f) \otimes (g_1) * ... * (g_n)$$

Consider the folowing functions where $p+q=n, m \geq 2$:

$$K^{p,q}_{n,m}(x,y,z) := (\frac{Im\tau}{\pi})^m \times$$

$$\sum_{\gamma_0+...+\gamma_m=0}' \frac{(x,\gamma_0)(y,\gamma_1+...+\gamma_{m-1})(z,\gamma_m) \cdot (\bar{\gamma}_m(\gamma_0-\gamma_1)+\gamma_m(\bar{\gamma}_0-\bar{\gamma}_1)) \cdot \gamma_m^{p-1} \cdot \bar{\gamma}_m^{q-1}}{|\gamma_0|^2 \cdot ... \cdot |\gamma_{m-1}|^2 |\gamma_m|^{2n}}$$

I will call them the generalized Eisenstein-Kronecker series. For $n=1$ this is the functions $K_{m+1}(x,y,z)$ defined in [G4], see also [G3]. For $n=1, m=2$ this function was considered by Deninger [D3].

Conjecture 8.3 *There exists a variation of mixed elliptic motives over $(E \times E \times E)/E$ such that its real periods are given by the generalized Eisenstein-Kronecker series.*

Define a homomorphism

$$K_{n,m} : \Big(\mathbb{Z}[(E \times E \times E)(\mathbb{C})]\Big)_E \longrightarrow Sym^n H^1(E(\mathbb{C}), \mathbb{C})$$

$$\{x,y,z\} \longmapsto \sum_{p+q=n} K^{p,q}_{n,m}(x,y,z)(dz)^p(d\bar{z})^q$$

Theorem 8.4 *Assume*

$$\text{Alt}_{(x_0,\dots,x_n)}\Big(\{f(x_0)\}_{m-1}\otimes f(x_0)\wedge g_1(x_1)\wedge\dots\wedge g_n(x_n)\Big)=0$$

in the group $\mathcal{B}_{m-1}(\mathbb{C}(E^{(n)}))\otimes\Lambda^{n+1}\mathbb{C}(E^{(n)})^*$. *Then*

$$r(\{f\}_m\otimes g_1\wedge\dots\wedge g_n)=K_{n,m}\circ\beta_{n,m}\Big(\{f\}_m\otimes g_1\circ\dots\circ g_n\Big)$$

The proof is completely similar to the proof of theorems 6.4 and theorems 3.4 and 4.7 in [G3], and thus is omitted.

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Max-Planck-Institute fur mathematik, Bonn, 53115, Germany; sasha@mpim-bonn.mpg.de

Department of Mathematics, Brown University, Providence, RI 02912, USA.