Integral geometry and differential equations

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To Iosif Bernstein for his 50-th birthday

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1 Introduction

The general problem. Let X be a smooth manifold of dimension n. Any n-form with compact support ω defines a functional on $C^\infty(X) : < \omega, f >= \int_X f\omega$.

Let $\mathcal{M}$ be a system of linear partial differential equations on one function on X. Denote by $Sol(\mathcal{M}, C^\infty(X))$ (or simply $Sol_{C^\infty}(\mathcal{M})$) the space of smooth solutions to $\mathcal{M}$.

Any n-form, of course, defines a linear functional on $Sol_{C^\infty}(\mathcal{M})$. However if the system $\mathcal{M}$ is not empty, the functional dimension of $Sol_{C^\infty}(\mathcal{M})$ is less then n and so many n-forms represent the same functional.

In this paper I adress the following questions:

Question 1. What is a *natural realization* for the (continuous) functionals on the space $Sol_{C^\infty}(\mathcal{M})$?

Question 2. What is a *natural realization* for the (continuous) linear maps from $Sol(\mathcal{M}, C^\infty(X))$ to $Sol(\mathcal{N}, C^\infty(Y))$?

I assume that a functional should have essentially one natural realization and it should be given in terms of the manifold X and system $\mathcal{M}$.

2. Natural functionals on solutions to $\mathcal{M}$: a very naive approach. Let $\mathcal{A}^m(X)$ be the space of smooth m-forms on X. Consider a linear differential operator

$$\kappa : C^\infty(X) \longrightarrow \mathcal{A}^m(X)$$

Definition 1.1 *κ is $\mathcal{M}$-**closed** if $d\kappa(f) = 0$ for any $f \in Sol_{C^\infty}(\mathcal{M})$.*

*κ is $\mathcal{M}$-**exact** if there exists a linear differential operator $\nu : C^\infty(X) \longrightarrow \mathcal{A}^{(m-1)}(X)$ such that $d\nu(f) = \kappa(f)$ for any $f \in Sol_{C^\infty}(\mathcal{M})$.*

*κ_1 is $\mathcal{M}$-**equivalent** to κ_2 if $\kappa_1(f) = \kappa_2(f)$ for any $f \in Sol_{C^\infty}(\mathcal{M})$.*

Consider the vector space

$$DSol(\mathcal{M})^m := \frac{\mathcal{M}-\text{equivalence classes of}\quad \mathcal{M}-closed\quad \kappa'\text{s}}{\mathcal{M}-\text{equivalence classes of}\quad \mathcal{M}-exact\quad \kappa'\text{s}}$$

Example . For zero system of differential equations this is $\mathcal{A}^n(X)$ when $m = n$ and zero otherwise (see also s 2.1 below).

Integrating closed m-form $\kappa(f)$ along a cycle γ we get a map

$$DSol(\mathcal{M})^m \otimes Sol(\mathcal{M}, C_0^\infty(X)) \otimes H^m(X, \mathbb{R}) \longrightarrow \mathbb{R} \tag{1}$$

A natural functional is a functional on $Sol_{C^\infty}(\mathcal{M})$ provided by a pair $[\kappa], [\gamma]$.

This definition was inspired by some examples in integral geometry ([GGS], [GGG]) as well as by the notion of conservation laws for nonlinear partial differential equations. Unlike for nonlinear equations, there is a general construction of elements in $DSol(\mathcal{M})^m$, (see (10) below).

3. Relation with integral geometry. Let B be a manifold of dimension m and a linear operator

$$I_K : C_0^\infty(B) \longrightarrow C^\infty(\Gamma) \qquad f(x) \longmapsto \int_B K(x,y)dx \tag{2}$$

is injective, transforms functions $f(x)$ to solutions of a linear system of PDE $\mathcal{M}$ on Γ, and $I_K(C_0^\infty(B))$ is dense in $Sol(\mathcal{M}, C^\infty(\Gamma))$.

Such a situation is typical in integral geometry. Namely, let $\{B_\xi\}$ be a family of submanifolds of a manifold B parametrized by a manifold Γ. Suppose on $\{B_\xi\}$ densities μ_ξ (depending smoothly on ξ) are given. Then there is an integral operator

$$I : C_0^\infty(B) \longrightarrow C^\infty(\Gamma) \qquad f(x) \longmapsto \int_{B_\xi} f(x)\mu_\xi$$

So here $K(x,y) = \mu(x,y) \cdot \delta(A)$ where $A := \{(x,\xi) | x \in B_\xi\} \subset B \times \Gamma$ is the incidence subvariety. Suppose I is injective. If $dim\Gamma > dimB$ the image of the integral transformation I is often satisfy a certain system of partial differential equations and, moreover, in many cases is characterized as the space of *all* solutions of this system (see [J], [GGS]).

Example. Let $f(x)$ be a smooth function in $\mathbb{R}^m$ and

$$I : f(x) \longmapsto If(y;r) := \int f(y + \omega \cdot r)d\omega$$

where $If(y;r)$ is the mean value of a function f over a radius r sphere centered at $y \in \mathbb{R}^m$. Then $If(y;r)$ satisfies the Darboux differential equation

$$\Big(\frac{\partial^2}{\partial r^2} - \sum_{i=1}^m \frac{\partial^2}{\partial y_i^2} + \frac{m-1}{r}\frac{\partial}{\partial r}\Big)If(y;r) = 0$$

For a point $x \in B$ the δ - functional $f \longmapsto f(x)$ can be considered as a functional on $Sol(\mathcal{M}, C^\infty(\Gamma))$. The problem of natural realization for this functional is just the problem of inversion of the integral transform I_K.

4. Content of the paper. In s.3 a class of natural linear maps between solution spaces is constructed. This provides a very general method for solving problems of integral geometry.

In s.4 we demonstrate how it works for the family of all spheres in $\mathbb{R}^m$. Our approach leads to *universal* inversion formulas which are *nonlocal* when m is odd and local when m is even. In particularly we show that a function f in $\mathbb{R}^m$ is determined by its mean values over spheres tangent to a given submanifold and derive explicit inversion formulas. The only known before case was the family of all spheres tangent to a plane (horospheres in the hyperbolic geometry, see [GGV]).

Applying our method to the system of PDE on the Grassmanninan describing the image of the Radon transform over k-planes we come to the "form κ" of Gelfand-Graev-Shapiro ([GGS],[GGGi]). Namely the latter is the residue of our universal inversion formula on the variety $\Gamma_x \subset \Gamma$ parametrized all submanifolds B_ξ passing through a given point $x \in B$ ([G2]).

The form κ appeared in [GGS] as a construction "ad hoc" and looks like a very special phenomena. In our approach the universal inversion formula is a very general property of the corresponding system of linear PDE. Its locality, however, is a rather rare phenomena, which generalizes the notion of lacunas for hyperbolic differential equations.

In particulary in these examples our natural functionals describe the whole dual to the space of solutions of a linear system of PDE.

The appropriate language for these problems is the language of $\mathcal{D}$-modules and derived categories. In s.3 we present a "naive" version of the story, where only the first nontrivial Ext-group is in the game. However, the degree of this "first" group is equal to the codimension of the characteristic variety of the system. This explanes why usually theory of PDE deals with m differential equations on m unknown functions: otherwise we will get Ext^k for $k > 1$. A more systematic presentation will appear in [G2].

In s.5 we introduce a bicategory of $\mathcal{D}$-modules, which is an algebraic model of integral geometry.

The objects of our bicategory are pairs $(X, \mathcal{M})$, where $\mathcal{M}$ is a (complex of) $\mathcal{D}$-module(s) on a variety X. A 1-morphism between $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ is the algebraic part of the data needed to construct a natural linear map $RHom_{\mathcal{D}}(\mathcal{M}, C^\infty(X)) \longrightarrow RHom_{\mathcal{D}}(\mathcal{N}, \mathcal{D}'(Y))$. Composition of 1-morphisms corresponds to the composition of natural linear maps. A 2-morphism between two 1-morphism reflects coinsidence of the the corresponding natural maps on functions.

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It is my pleasure to dedicate this paper to Iosif Bernstein in the occasion of his 50-th birthday.

2 The Green class of a $\mathcal{D}$-module

1. Language of $\mathcal{D}$-modules. Let $\mathcal{D}_X$ (or $\mathcal{D}$) be the sheaf of rings of differential operators on a manifold X. Suppose we have a linear system $\mathcal{M}$ of p differential equations on q functions $f_1, ..., f_q$, $\mathcal{M} = \{\sum_{j=1}^q d_{ij} f_j = 0, i = 1, ..., p\}$. Then we can assign to $\mathcal{M}$ a coherent $\mathcal{D}$-module $\mathcal{M}$ given by q generators $e_1, ..., e_q$ and p relations:

$$\mathcal{M} = \frac{\oplus \mathcal{D} \cdot e_i}{+\mathcal{D}(\sum d_{ij} f_j)}$$

From the other hand a coherent $\mathcal{D}$-module $\mathcal{M} = Coker(\mathcal{D}^p \longrightarrow \mathcal{D}^q)$ provides us with a linear system of p differential equations on q functions. In this language a solution f to the system $\mathcal{M}$ in some space of functions $\mathcal{F}$ is nothing else then a morphism of $\mathcal{D}$-modules $\alpha_f : \mathcal{M} \longrightarrow \mathcal{F}$.

The De Rham complex $DR(\mathcal{M})$ of a $\mathcal{D}$-module $\mathcal{M}$ is defined as follows:

$$\mathcal{M} \longrightarrow \Omega^1 \otimes_{\mathcal{O}} \mathcal{M} \longrightarrow ... \longrightarrow \Omega^{n-1} \otimes_{\mathcal{O}} \mathcal{M} \longrightarrow \Omega^n \otimes_{\mathcal{O}} \mathcal{M} \tag{3}$$

where $\Omega^n \otimes_{\mathcal{O}} \mathcal{M}$ is sitting in degree 0, d has degree +1 and $d(m \otimes \omega) := m \otimes d\omega + \sum \frac{\partial}{\partial x_i} m \otimes dx_i \wedge \omega$ (it does not depend on coordinates x_i) and $\mathcal{O}$ is the structural sheaf of X.

Let $\mathcal{A}^k(X)$ be the space of C^∞ k-forms on X. The C^∞-De Rham complex $DR(\mathcal{M} \otimes_{\mathcal{O}} C^\infty(X))$ of a $\mathcal{D}$-module $\mathcal{M}$ which looks as follows:

$$\mathcal{M} \longrightarrow \mathcal{A}^1 \otimes_{\mathcal{O}} \mathcal{M} \longrightarrow ... \longrightarrow \mathcal{A}^{n-1} \otimes_{\mathcal{O}} \mathcal{M} \longrightarrow \mathcal{A}^n \otimes_{\mathcal{O}} \mathcal{M} \tag{4}$$

One can show that $DSol(\mathcal{M})^m = H^{m-n}DR(\mathcal{M} \otimes_{\mathcal{O}} C^\infty(X))$.

For example using the Koszule complex one can see that $DSol(\mathcal{D}_X)^n = \mathcal{A}^n(X)$ and $DSol(\mathcal{D}_X)^m = 0$ for $m < n$.

Notice that $DR(C^\infty(X))$ coincides with $\mathcal{A}^\bullet(X)[n]$, the C^∞-de Rham complex of X shifted by n to the left. Therefore any $f \in Sol(\mathcal{M}, C^\infty(X))$ defines a homomorphism of complexes

$$DR(\mathcal{M} \otimes_{\mathcal{O}} C^\infty(X)) \xrightarrow{\tilde{f}} DR(C^\infty(X) \otimes_{\mathcal{O}} C^\infty(X)) \xrightarrow{m} \mathcal{A}^\bullet(X)[n]$$

Here m is induced by the homomorphism of $\mathcal{D}$-modules $C^\infty(X) \otimes_{\mathcal{O}} C^\infty(X) \longrightarrow C^\infty(X)$.

2. The duality functor ([Be], [Bo]). Ω_X has canonical structure of a right $\mathcal{D}_X$-module given by the formula $\omega \cdot f = f\omega, \quad \omega \cdot \xi := -L_\xi \omega$ where $f \in \mathcal{O}_X$ and ξ is a vector field. Set

$$\mathcal{D}_X^\Omega := \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X^{-1} = Hom_{\mathcal{O}_X}(\Omega_X, \mathcal{D}_X^r) \tag{5}$$

where $\mathcal{D}_X^r$ is $\mathcal{D}_X$ viewed as a right $\mathcal{D}$-module via right multiplication. Then (5) carries 2 commuting left $\mathcal{D}_X$-modules structures. The first one is provided by the left multiplication on $\mathcal{D}_X$, and the second is given by the rule $\xi \circ (\lambda)(\omega) = \lambda(\omega \cdot \xi) - \lambda(\omega) \cdot \xi$ where ξ is a vector field and $\lambda \in Hom_{\mathcal{D}_X}(\Omega_X, \mathcal{D}_X^r)$.

For any coherent $\mathcal{D}_X$-module $\mathcal{M}$ the second structure provides the structure of left $\mathcal{D}_X$-module on sheaves $Ext^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega)$.

Accoding to the Roos theorem for a $\mathcal{D}$-module $\mathcal{M}$

$$\text{if} \quad codim S.S.(\mathcal{M}) = k \quad \text{then} \quad Ext^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega) = 0 \quad \text{for} \quad i < k. \tag{6}$$

Let $D_{coh}(\mathcal{D}_X)$ be the derived category of bounded complexes of $\mathcal{D}_X$-modules considered modulo quasiisomorphisms whose cohomology groups are coherent $\mathcal{D}_X$-modules. Objects of $D_{coh}(\mathcal{D}_X)$ will be denoted $\mathcal{M}^\bullet$.

Let us define duality $\star : D_{coh}(\mathcal{D}_X)^0 \longrightarrow D_{coh}(\mathcal{D}_X)$ by

$$\star\mathcal{M}^\bullet := RHom_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X^\Omega)[dimX] \tag{7}$$

In particulary $H^i(\star\mathcal{M}^\bullet) = Ext^{dimX+i}_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X^\Omega)$. To compute $\star\mathcal{M}^\bullet$ we should find a bounded complex

$$\mathcal{P}^\bullet = \{\longrightarrow \mathcal{P}^{-2} \longrightarrow \mathcal{P}^{-1} \longrightarrow \mathcal{P}^0 \longrightarrow ...\}$$

of locally projective coherent $\mathcal{D}$-modules quasiisomorphic to $\mathcal{M}^\bullet$ and set $\star\mathcal{M}^\bullet = \star\mathcal{P}^\bullet$ where $(\star\mathcal{P})^i = \star(\mathcal{P}^{-dimX-i}) := Hom_{\mathcal{D}_X}(\mathcal{P}^{-dimX-i}, \mathcal{D}_X^\Omega)$.

3. The Green class of $\mathcal{M}^\bullet$. For any $\mathcal{D}$-modules $\mathcal{M}$ and $\mathcal{N}$ the tensor product $\mathcal{N} \otimes_{\mathcal{O}} \mathcal{N}$ has canonical $\mathcal{D}$-module structure where the vector fields acts by the Leibniz rule.

Theorem 2.1 *Let $\mathcal{M}^\bullet \in D^b_{coh}(\mathcal{D}_X)$ and $\mathcal{N}^\bullet \in D^b(\mathcal{D}_X)$. Then one has canonical isomorphism in the derived category of sheaves on X functorial with respect to $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$*

$$DR(\star\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{N}^\bullet)[-dimX] = RHom_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \tag{8}$$

The proof see for example in ch. VII, 9.7, [Bo].

Placing to (8) $\mathcal{N} = C^\infty(X)$ and using $\star\star\mathcal{M} = \mathcal{M}$ we get

$$DR(\mathcal{M} \otimes_{\mathcal{O}} C^\infty(X))[-dimX] = RHom_{\mathcal{D}_X}(\star\mathcal{M}, C^\infty(X)) \tag{9}$$

One can rewrite (9) as:

$$DSol(\mathcal{M})_m = RHom^{(m-dimX)}_{\mathcal{D}_X}(\star\mathcal{M}, C^\infty(X)) \tag{10}$$

The identity map $Id \in Hom_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{M}^\bullet)$ provides canonical element

$$G_{\mathcal{M}^\bullet} \in H^{dimX}\Big(DR(\star\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{M}^\bullet)\Big) \tag{11}$$

I will call it *the Green class of* $\mathcal{M}^\bullet$. Any solutions

$$\varphi \in RHom_{\mathcal{D}}(\mathcal{M}^\bullet, C^\infty(X)) \quad \text{and} \quad v \in RHom_{\mathcal{D}}(*\mathcal{M}^\bullet, C^\infty(X))$$

provides a morphism

$$DR\Big(*\mathcal{M}^\bullet \otimes_{\mathcal{O}} \mathcal{M}\Big) \xrightarrow{\bar\varphi \otimes \bar v} DR\Big(C^\infty(X) \otimes_{\mathcal{O}} C^\infty(X)\Big) \xrightarrow{m} \mathcal{A}^\bullet(X)$$

It sends the Green class of $\mathcal{M}^\bullet$ to a cohomology class $[G_{\mathcal{M}^\bullet}(\varphi, v)]$ on X.

4. Relation with the classical Green formula. Let P be a differential operator. Set

$$\mathcal{P} = \frac{\mathcal{D}_X}{\mathcal{D}_X \cdot P} \quad \text{and} \quad \mathcal{P}^* = \frac{\mathcal{D}_X \otimes \Omega_X^{-1}}{P \cdot \mathcal{D}_X \otimes \Omega_X^{-1}}$$

Here $\mathcal{D}_X \otimes \Omega_X^{-1}$ is considered as a left $\mathcal{D}$-module with respect to the second structure. Notice that $Hom_{\mathcal{D}}(\mathcal{D}_X \otimes \Omega_X^{-1}, C^\infty(X)) = \mathcal{A}^n(X)$. Let $v \in \mathcal{A}^n(X)$. According to the Green formula there exists an $(n-1)$-form $\omega_{n-1}(\varphi; P; v)$ on X such that

$$P\varphi \cdot v - \varphi \cdot P^* v = d\omega_{n-1}(\varphi; P; v)$$

Lemma 2.2 *a)* $\star\mathcal{P}$ *is isomorphic to* $\mathcal{P}^*[1]$.

b) The class $[G_{\star\mathcal{P}}(\varphi, v)] \in H^{n-1}(\mathcal{A}^\bullet(X))$ *defined by smooth solutions* $\varphi \in Sol(\mathcal{P})$ *and* $v \in Sol(\mathcal{P}^*)$ *coincides with the cohomology class of the* $(n-1)$*-form* $\omega_{n-1}(\varphi; P; v)$.

Remark. There is canonical involution on $(\mathcal{D}_X^r \otimes \Omega_X^{-1})$ interchanging the left $\mathcal{D}_X$-structures. It sends $P \cdot \omega^{-1}$ just to $P^t \cdot \omega^{-1}$ where P^t is the transposed to P defined using the form ω.

4. The Green formula and the Bar construction. (The constructions in this subsection were also considered by M.M.Kapranov). Let E^1 and E^2 be vector bundles over an n-dimensional manifold X and $E^1 \xrightarrow{P} E^2$ be a differential operator. Set $V_i := E^{i*} \otimes \mathcal{A}^n$. There are canonical pairings

$$\Gamma_0(X, E^i) \otimes \Gamma(X, V_i) \longrightarrow \mathbb{R} \qquad (\varphi, g \otimes \omega) \longrightarrow \int_X (\varphi, g)\omega$$

So one has the adjoint operator $V_1 \xleftarrow{P^*} V_2$. It is a differential operator of the same order as P uniquely defined by the property $(\varphi_1, P^* v_2) = (P\varphi_1, v_2)$.

Now suppose we have a sequence (not necessarily a complex) of differential operators

$$E^0 \xrightarrow{P_1} E^1 \xrightarrow{P_2} ... \xrightarrow{P_{k-1}} E^{k+1}$$

Consider the sequence of adjoint differential operators

$$V_0 \xleftarrow{P_1^*} V_1 \xleftarrow{P_2^*} ... \xleftarrow{P_k^*} V_{k+1}$$

Theorem 2.3 *For any* k *there exists forms* $\omega_{n-k}(\varphi_0; P_1, ..., P_k; v_k)$ *satisfying the conditions*

$$d\omega_{n-k}(\varphi_0; P_1, ..., P_k; v_k) = \omega_{n-k+1}(P_1\varphi_0; P_2, ..., P_{k+1}; v_k) +$$

$$\sum_{i=1}^{k-1} (-1)^i \omega_{n-k+1}(\varphi_0; P_1, ..., P_i \circ P_{i+1}, ..., P_k; v_k) + (-1)^k \omega_{n-k+1}(\varphi_0; P_1, ..., P_{k-1}; P_k^* v_k)$$

where we formally set $\omega_n(\varphi; 1; v) := \varphi \cdot v$

5 How to compute the Green class. Let us call a $\mathcal{D}$-module $\mathcal{M}$ *excellent* if the object $\star\mathcal{M}$ is concentrated in just one degree, i.e. $H^i(\star\mathcal{M}) = 0$ for all i but one. By the Roose theorem (see 6 or [Be], [Bo]) this degree is necessarily $-d_{\mathcal{M}}$. In this case set $\tilde{\star}\mathcal{M} := H^{-d_{\mathcal{M}}}(\star\mathcal{M})$. Consider a locally free resolution of a $\mathcal{D}$-module $\mathcal{M}$:

$$\mathcal{P}^\bullet = \{\mathcal{P}^{-k} \longrightarrow ... \longrightarrow \mathcal{P}^{-2} \longrightarrow \mathcal{P}^{-1} \longrightarrow \mathcal{P}^0\}$$

Let

$$*\mathcal{P}^\bullet = \{*(\mathcal{P}^0) \longrightarrow *(\mathcal{P}^1) \longrightarrow *(\mathcal{P}^2) \longrightarrow ... \longrightarrow *(\mathcal{P}^k)\}[d_X]$$

be the dual complex. Then $E^\bullet := Hom_{\mathcal{D}}(\mathcal{P}^\bullet, C^\infty(X))$ is a complex of differential operators between vector bundles:

$$E^0 \xrightarrow{P_1} E^1 \xrightarrow{P_2} ... \xrightarrow{P_k} E^k$$

The adjoint complex

$$V_\bullet := \quad \{V_k \xrightarrow{P_k^*} V_{k-1} \xrightarrow{P_{k-1}^*} ... \xrightarrow{P_1^*} V_0\}$$

is canonically isomorphic to $Hom_{\mathcal{D}}(*\mathcal{P}^\bullet, C^\infty(X))[-d_X - k]$.

Let us suppose that $\mathcal{D}$-module $\mathcal{M}$ is excellent and moreover admits a locally free resolution of the minimal possible length $k = d_{\mathcal{M}}$. (This is usually the case in integral geometry). Then

$$Sol(\mathcal{M}, C^\infty(X)) = KerP_1 \quad \text{and} \quad Sol(\tilde{*}\mathcal{M}, C^\infty(X)) = KerP_k^*$$

Therefore for any $\varphi_0 \in KerP_1$ and $v_k \in KerP_k^*$ the differential form $\omega_{n-d_{\mathcal{M}}}(\varphi_0; \mathcal{P}^\bullet; v_k)$ is closed.

Theorem 2.4 *The cohomology class of the form $\omega_{n-d_{\mathcal{M}}}(\varphi_0; \mathcal{P}^\bullet; v_k)$ coinsides with the Green class $G_{\mathcal{M}}(\varphi_0; v_k)$.*

3 Integral geometry: the general scheme

Let $\mathcal{M}$ and $\mathcal{N}$ be excellent $\mathcal{D}$-modules on manifolds X and Y (see s.2.5), so $\tilde{*}\mathcal{M} := (*\mathcal{M})[-d_{\mathcal{M}}]$ is a $\mathcal{D}$-module. Let $c_{\mathcal{M}}$ be the codimension of the singular support of $\mathcal{M}$ in T^*X, so $d_{\mathcal{M}} + c_{\mathcal{M}} = dim(X)$. Then solutions

$$f \in Hom_{\mathcal{D}}(\mathcal{M}, C^\infty(X)) \quad \text{and} \quad g \in Hom_{\mathcal{D}}(\tilde{*}\mathcal{M}, C^\infty(X))$$

provides a homomorphism

$$H^{c_{\mathcal{M}}} DR\Big(\tilde{*}\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}\Big) \xrightarrow{\bar{f} \otimes \bar{g}} H^{c_{\mathcal{M}}} DR\Big(C^\infty(X) \otimes_{\mathcal{O}} C^\infty(X)\Big) \xrightarrow{m} H^{d_{\mathcal{M}}} \mathcal{A}^\bullet(X)$$

The Green class of $\mathcal{M}$ goes under this map to a cohomology class of degree $d_{\mathcal{M}}$ on X. Recall that we put $DR(\mathcal{M})$ in degrees $[-dim(X), 0]$, while the smooth de Rham complex $\mathcal{A}^\bullet(X)$ is sitting in degrees $[0, dim(X)]$.

Let us define a natural linear map

$$I: \quad Sol(\mathcal{M}, C^\infty(X)) \longrightarrow Sol(\mathcal{N}, \mathcal{D}'(Y)) \tag{12}$$

by a kernel

$$K_I(x, y) \in Sol(\tilde{\star}\mathcal{M} \boxtimes \mathcal{N}, \mathcal{D}'(X \times Y)) \tag{13}$$

and a cycle γ_X of dimension $d_{\mathcal{M}}$ in X as follows. Let $\tilde{G}_{\mathcal{M}}(\cdot, \cdot)$ be a cocycle in $DR\Big(\tilde{*}\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}\Big)$ representing the Green class. Using solutions $K_I(x, y)$ of $\tilde{\star}\mathcal{M}$ (where y is considered as a parameter) and $f(x)$ of $\mathcal{M}$ we get a closed differential form $\tilde{G}_{\mathcal{M}}(K_I(x, y), f(x))$ of degree $d_{\mathcal{M}}$ on X. Set

$$f(x) \longmapsto \int_\gamma \tilde{G}_{\mathcal{M}}(K_I(x, y), f(x)) \in Sol(\mathcal{N}, \mathcal{D}'(Y)) \tag{14}$$

Under certain assumption on the wave front of the kernel $K_I(x,y)$, which we will assume below, the integral over cycle γ makes sence and the image of (13) lies in $C^\infty(Y)$.

Then a (natural) inverse for I is an integral transformation

$$J: \quad Sol(\mathcal{N}, C^\infty(Y)) \longrightarrow Sol(\mathcal{M}, C^\infty(X)) \tag{15}$$

$$J: \varphi(x) \longmapsto \int_{\gamma_Y} \tilde{G}_{\mathcal{N}}(K_J(x,y), \varphi(y)) \tag{16}$$

defined by a certain $d_{\mathcal{N}}$-cycle γ_Y in Y and a kernel

$$K_J(x,y) \in Sol(\mathcal{M} \boxtimes \tilde{\star}\mathcal{N}, \mathcal{D}'(X \times Y)) \tag{17}$$

This data defines also a transformation

$$J^t: \quad Sol(\tilde{\star}\mathcal{M}, C^\infty(X)) \longrightarrow Sol(\tilde{\star}\mathcal{N}, C^\infty(Y)) \tag{18}$$

$$g(x) \longmapsto \int_{\gamma_X} \tilde{G}_{\mathcal{M}}(g(x), K_J(x,y)) \tag{19}$$

There is canonical pairing

$$< \cdot, \cdot >_{\mathcal{M}} \quad : \quad Sol(\tilde{\star}\mathcal{M}, C^\infty(X)) \otimes Sol(\mathcal{M}, C^\infty(X)) \otimes H_{d_{\mathcal{M}}}(X, \mathbb{R}) \longrightarrow \mathbb{R} \tag{20}$$

$$< g, f >_{\mathcal{M}} \quad := \int_{\gamma_X} \tilde{G}_{\mathcal{M}}(g(x), f(x))$$

and a similar one for $\mathcal{N}$.

Theorem 3.1 *(the Plancherel formula) . Let J be a natural inverse for I: $J \circ I = id_X$. Then for $f \in Sol(\mathcal{M}, C^\infty(X)), g \in Sol(\tilde{\star}\mathcal{M}, C^\infty(X))$ one has*

$$< g, f >_{\mathcal{M}} \quad = \quad < J^t g, If >_{\mathcal{N}} \tag{21}$$

Proof. $< g, f >_{\mathcal{M}} \quad = \quad < g, J \circ If >_{\mathcal{N}}$. So the theorem follows from

Lemma 3.2 *Let $\varphi \in Sol(\mathcal{N}, C^\infty(Y))$ and $g \in Sol(\tilde{\star}\mathcal{M}, C^\infty(X))$. Then*

$$< g, J\varphi >_{\mathcal{M}} \quad = \quad < J^t g, \varphi >_{\mathcal{N}} \tag{22}$$

Proof. The Green class is multiplicative with respect to the $\boxtimes$ - product. So we can set $\tilde{G}_{\mathcal{M}\boxtimes\mathcal{N}} := \tilde{G}_{\mathcal{M}} \boxtimes \tilde{G}_{\mathcal{N}}$. Consider the following solutions

$$g(x) \boxtimes \varphi(y) \in Sol(\tilde{\star}\mathcal{M} \boxtimes \tilde{\star}\mathcal{N}, C^\infty(X \times Y))$$

$$K_J(x,y) \in Sol(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{D}'(X \times Y))$$

They are solutions to the dual systems. So there is a pairing

$$< g(x) \boxtimes \varphi(y), K_J(x,y) >_{\mathcal{M}\boxtimes\mathcal{N}} \tag{23}$$

related to the cycle $\gamma_X \times \gamma_Y$. We can evaluate it computing first the pairing along X and then along Y. In this case we get the right- hand side of (22). Computing first pairing along Y and then along X we get the left-hand side of (22).

The kernel K_J is a much more simple (and fundamental) object then the actual integral transformation J. The reasons are the following:

1) The kernel K_J is a canonically defined distribution, while the formula for $J\varphi(x)$ depends on a cocycle $\tilde{G}_{\mathcal{N}}$ representing the Green class.

2) Explicit calculation of cocycle $\tilde{G}_{\mathcal{N}}$ can be a nontrivial problem and so the final formula for the right-hand side of (16) could be quite complicated even for a very simple kernel K_J.

So the problem of inversion of the transformation I splits on 3 steps:

Step 1. Find a distribution (17).

Step 2. Compute a cocycle $\tilde{G}_{\mathcal{N}}$ for the Green class.

Step 3. Find a cycle γ_Y.

The distribution (17) should be uniquely defined if exist. However it may not exist.

The Green class always exist. Different cocycles representing it together with different choices of cycles γ_Y provides the diversity of concrete inversion formulas.

4 Integral geometry on families of spheres and intertwiners

1. Integral transformation. Let

$$S^m = \{x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2 = 0\}/\mathbb{R}^*$$

be a sphere in $\mathbb{R}P^{m+1}$. The stereographic projection identifies the set of its hyperplane sections with the family of all spheres in $\mathbb{R}^m$.

Let $Q_{m+1} := \{x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2 = 0\}$ be a cone in $\mathbb{R}^{m+2} \backslash 0$. It has two connected components: Q_{m+1}^+ in the half space $x_{m+2} > 0$ and its opposite Q_{m+1}^-. Denote by $\Phi_\lambda(S^m)$ the space of all homogeneous functions of degree λ on the cone Q_{m+1}^+.

Let β_m be a hyperplane section of Q_{m+1}^+ isomorphic to a sphere. The orientation of $\mathbb{R}^{m+2}$ provides canonical orientation of β_m. Namely, β_m inside of the cone is cooriented out of the origin , and the cone itself has canonical coorientation (outside of the convex component) in $\mathbb{R}^{m+2}$. Let β_m^+ be an oriented this way cycle. Its homology class is a generator of $H_m(Q_{m+1}^+, \mathbb{Z})$. Set

$$\sigma_{m+2}(x, dx) := \sum_{i=1}^{m+2} (-1)^{i-1} x_i dx_1 \wedge ... \hat{dx_i} ... \wedge dx_{m+2}$$

There is canonical nondegenerate pairing

$$< \cdot, \cdot >_{S^m} : \Phi_{1-m}(S^m) \otimes \Phi_{-1}(S^m) \to \mathbb{R} \tag{24}$$

$$< f, g >_{S^m} = \int_{\beta_m^+} \delta(x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2) f(x) g(x) \sigma_{m+2}(x, dx) \tag{25}$$

Here we integrate the closed m-form on Q_{m+1}^+. By definition it is the restriction to Q_{m+1}^+ of any form α_m satisfying the condition

$$d(x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2) \wedge \alpha_m = f(x) g(x) \sigma_{m+2}(x, dx)$$

The restriction is well defined on Q_{m+1}^+.

Let $\xi_1, ..., \xi_{m+2}$ be coordinates in $(\mathbb{R}^{m+2})'$ dual to x_i and $< \xi, x > = \sum \xi_i x_i$. For $f \in \Phi_{1-m}(S^m)$ set

$$(If)(\xi) = \int_{\beta_m^+} \delta(x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2) f(x) \delta(< \xi, x >) \sigma_{m+2}(x, dx) \tag{26}$$

Consider the following kernel:

$$K_{-(m-1)}(\xi, x) := \delta^{(m-2)}(< \xi, x >) \quad \textit{for odd } \mathrm{m} \textit{ and } \quad < \xi, x >^{-(m-1)} \quad \textit{for even } \mathrm{m}$$

It defines an integral transformation acting on $g \in \Phi_{-1}(S^m)$ as follows:

$$(J^t) g(\xi) = \int_{\beta_m^+} \delta(x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2) g(x) K_{-(m-1)}(< \xi, x >) \sigma_{m+2}(x, dx) \tag{27}$$

Set

$$\Delta := \partial^2_{\xi_1} + ... + \partial^2_{\xi_{m+1}} - \partial^2_{\xi_{m+2}}; \qquad L_a := \sum_{i=1}^{m+2} \xi_i \partial_{\xi_i} - a$$

Let $\mathcal{N}_a$ be the following system of differential equations

$$\mathcal{N}_a : \qquad L_a\varphi = 0, \quad \Delta\varphi = 0$$

Lemma 4.1 *a) $If(\xi)$ is an even function satisfying the system $\mathcal{N}_{-1}$.*
b) $(J^t)\varphi(\xi)$ is an odd function satisfying the system $\mathcal{N}_{-(m-1)}$.

2. The Green class. Now let us make the crucial step. Consider the following m-form:

$$\omega_m(v;\varphi) := \tag{28}$$

$$\sum_{1\le i<j\le m+2} (-1)^{i+j-1}\Big(\xi_i \cdot \varepsilon_j(v \cdot \varphi'_{\xi_j} - v'_{\xi_j} \cdot \varphi) - \xi_j \cdot \varepsilon_i(v \cdot \varphi'_{\xi_i} - v'_{\xi_i} \cdot \varphi)\Big) d\xi_1 \wedge ...\hat{d\xi_i}...\hat{d\xi_j}... \wedge d\xi_{m+2}$$

Here $\varepsilon_{m+2} = -1$ and $\varepsilon_j = 1$ if $j \neq m+2$.

Let $\omega_{m+1}(v;\Delta;\varphi)$ be the Green form of the Laplacian Δ:

$$\omega_{m+1}(v;\Delta;\varphi) = \sum_{1\le j\le m+2} (-1)^{j-1}\varepsilon_j(\varphi_{\xi_j} \cdot v - \varphi \cdot v_{\xi_j}) d\xi_1 \wedge ...\hat{d\xi_j}... \wedge d\xi_{m+2} \tag{29}$$

Then (28) is the contraction of the Green form (29) with the Euler vector field L.

$$\omega_m(v;\varphi) = -\frac{1}{2} i_L \omega_{m+1}(v;\Delta;\varphi)$$

Theorem 4.2 *a) $\circledast\mathcal{N}_a = \mathcal{N}_b$ where $a + b + m = 0$.*
b) The form $\omega_m(\varphi;v)$ represents the Green class $G_{\mathcal{N}_a}(\varphi;v)$ of the system $\mathcal{N}_a$.

In particulary the form $\omega_m(\varphi;v)$ is closed if the functions v and φ are solutions of the systems $\mathcal{N}_a$ and $\mathcal{N}_b$ where $a + b + m = 0$.

Scetch of the proof. Consider a complex of $\mathcal{D}$ - modules $\mathcal{D} \xrightarrow{d} \mathcal{D}^2 \xrightarrow{d} \mathcal{D}$ sitting in degrees [-2,0] (d has degree +1) that we will visualize as follows:

$$\begin{array}{ccccc} & & \mathcal{D} & & \\ & {\scriptstyle\Delta}\nearrow & & \searrow{\scriptstyle L_a} & \\ \mathcal{D} & & & & \mathcal{D} \\ & {\scriptstyle -L_{a-2}}\searrow & & \nearrow{\scriptstyle\Delta} & \\ & & \mathcal{D} & & \end{array}$$

One can prove that it is a resolution of the $\mathcal{D}$- module $\mathcal{N}_a$ and a) follows easyly. To calculate the Green class we use theorem (2.4) for this resolution .

Remark. More generally, for any homogeneous differential operator P with constant coefficients in $\mathbb{R}^n$ the Green form for the system $Pf = 0, L_af = 0$ is equal to $-\frac{1}{2}i_L\omega_{n-1}(v;P;\varphi)$.

3. Construction of the inverse operator. The function $If(\xi)$ is defined in the domain $\tilde{\Gamma} := \{\xi | \xi_1^2 + ... + \xi_{m+1}^2 > \xi_{m+2}^2\}$. Let $\Gamma = \tilde{\Gamma}/\mathbb{R}^*_+$ be the manifold of all oriented rays inside $\tilde{\Gamma}$. Its closure $\bar{\Gamma}$ parametrizes *oriented* hyperplane sections of the sphere S^m.

$\Gamma = S^{m+1} \backslash \mathcal{D}_+ \cup \mathcal{D}_-$ where $\mathcal{D}_+$ is a ball $\{\xi_1^2 + ... + \xi_{m+1}^2 < \xi_{m+2}^2\}/(\mathbb{R}^*)^+$ and $\mathcal{D}_- = -\mathcal{D}_+$. Therefore $H_m(\Gamma,\mathbb{Z}) = \mathbb{Z}$. Consider the cycle γ_m of rays in the hyperplane $\xi_{m+2} = 0$. It is cooriented in the direction to the ball $\mathcal{D}_+$). So an orientation of $\mathbb{R}^{m+2}$ provides an orientation of this cycle. Denote by γ_m^+ the oriented cycle. Its homology class is a generator of $H_m(\Gamma,\mathbb{Z})$.

Lemma 4.3 *The form $\omega_m(\varphi; v)$ can be pushed down to Γ*

Let K be a a compact hypersurface in Γ. Its homology class $[K] \in H_n(\Gamma)$ is equal to $d(K) \cdot [\gamma_M^+]$. The integer $d(K)$ is the intersection number of the class $[K]$ with the "Euler" class consisting of spheres passing through a given point $x \in S^m$ and tangent to a given hyperplane in $T_x S^m$.

Let C be a submanifold in S^m. Consider the family Γ_C of oriented hyperplane sections of the sphere S^m tangent to C. For example when C is a point $d(\Gamma_C) = 1$.

Theorem 4.4 *a) For any m-cycle $K \in \Gamma$ one has*

$$d(K) \cdot < f, g >_{S^m} = c_m \cdot \int_K \omega_m\Big(If; J^t g\Big)$$

where $-c_m = \frac{(-1)^{(m-1)/2}}{(2\pi)^{m-1}}$ *for odd m and* $\frac{(-1)^{m/2}(m-1)!}{(2\pi)^m}$ *for even m.*

b) In particulary

$$d(K) \cdot f(x) = c_m \cdot \int_K \omega_m\Big(If; K_{-(m-1)}(\xi, x)\Big) \tag{30}$$

So the inversion formula is local for odd m and nonlocal for even m.

Proof. Let $n = (0 : .. : 0 : 1 : 1)$ be the "North pole"in S^m. The variety Γ_n parametrizing the hyperplane sections of the sphere S^m passing through the point n is a hyperplane given by equation $\xi_{m+1} + \xi_{m+2} = 0$.

It is sufficient to prove these formulas for one cycle K. The following lemma shows that for $K = \Gamma_n$ they reduce to the Plancherel theorem and the inversion formula for the classical Radon transform (see [GGG]). Set $\xi' = (\xi_1, ..., \xi_{m+1})$.

Lemma 4.5 . *The restriction of the form $\omega_m(\varphi; v)$ to Γ_n is equal to*

$$\omega_m(\varphi; v)|_{\Gamma_n} = (-1)^{m+1}\Big(v \cdot (\partial_{\xi_{m+1}} - \partial_{\xi_{m+2}})\varphi - \varphi \cdot (\partial_{\xi_{m+1}} - \partial_{\xi_{m+2}})v\Big)\sigma_m(\xi', d\xi')$$

Integrating by parts we get $2 \cdot (-1)^m\Big(\varphi \cdot (\partial_{\xi_{m+1}} - \partial_{\xi_{m+2}})v\Big)\sigma_m(\xi', d\xi')$

4. Admissible families of spheres. Restricting our integral operator I to a family K of spheres we get an integral transformation $I_K : \Phi_{1-m}(S^m) \longrightarrow \Psi^+_{-1}(K)$. Here $\Psi^+_\lambda(\Gamma_C)$ is the space of even homogeneous degree λ functions on $\tilde{K} \subset \mathbb{R}^{m+2}$.

Apriory the restriction of the form $\omega_m(\varphi; v)$ to a hypersurface K depends not only on the restriction of the functions φ and v on K, but on their first derivatives in the normal direction to K. Therefore for general K the right hand side of (30) can not be computed if know only $I_K(f)$. So it does not give an inversion formula for the integral transformation I_K.

Definition 4.6 *A hypersurfaces $K \subset \Gamma$ is called admissible if the cohomology class of the restriction of the form $\omega_m(\varphi; v)$ to K depends only on the restrictions of smooth solutions $\varphi \in Sol_{C^\infty}(\mathcal{N}_{-1})$, $\quad v \in Sol_{C^\infty}(\mathcal{N}_{-1})$ to K.*

More precisely, this means that there exists a bidifferential operator $\nu : C^\infty(K)^{\otimes 2} \longrightarrow \mathcal{A}^m(K)$ such that for any φ, v as above $\quad [\omega_m(\varphi; v)|_K] = [\nu(\varphi|_K, v|_K)]$.

Theorem 4.7 *a) For any $C \subset S^m$ the hypersurface Γ_C is admissible.*

b) Any admissible hypersurface in Γ is a piece of a hypersurface Γ_C for a certain $C \subset S^m$.

Proof of a). For $C = n$ this follows from the lemma (4.5). Indeed, the vector field $(\partial_{\xi_{m+1}} - \partial_{\xi_{m+2}})$ is tangent to the hyperplane Γ_n.

In general we proceed as follows. The form $\omega_m(\varphi; v)$ is given by a bidifferential operator of first order (see (28), so its restriction to K is determined by the restriction of the functions φ and v to the

1-st infinitesimal neighborhood of K. Let $\eta \in \Gamma_C$ and $t(\eta)$ be the tangency point of the hyperplane $< \eta, x >= 0$ with C. Then one can see that the tangent space to Γ_K at a point $t(\eta)$ coinsides with $\Gamma_{t(\eta)}$.

5. Inversion of the integral transform related to an admissible family. The restriction of the form $\omega_m(If; K_{-(m-1)}(\xi, x))$ to Γ_C depends only $I_{\Gamma_C} f$. So one can expect the inversion formula

$$d(\Gamma_C)f(x) = c_m \cdot \int_{\Gamma_C} \omega_m\Big(I_{\Gamma_C} f; K_{-(m-1)}(\xi, x)\Big) \tag{31}$$

similar to (30). However the cycle Γ_C lies in the closure $\bar{\Gamma}$, while the form $\omega_m(\varphi; v)$ was well defined only inside Γ, so it is apriory unclear whether the formula makes sence and is it possible to use the Stokes theorem.

To avoid this trouble we consider the integral transformation I_C only on the subspace of functions vanishing in a small neighborhood of the subvariety C in S^m. Assuming this let us perturbate the cycle Γ_C near the boundary of Γ by moving it a little bit inside of Γ.

Geometrically this means that we replace small spheres tangent to C by close to them small spheres which are not tangent to C. This perturbation does not affect the integral (31). Indeed, if a function $f(x)$ vanishes in a small neighborhood of C its integrals over the spheres in this neighborhood are equal to zero.

Remark. The cycle K becomes homologous to 0 in the sphere S^{m+1} parametrizing *all* oriented hyperplanes.

6. The general intertwiners. Let $\Phi^+_\lambda(S^m)$ (resp. $\Phi^-_\lambda(S^m)$) be the set of all even (odd) homogeneous functions of degree λ on the cone Q_{m+1}

Let $\varepsilon(1)$ the 1-dimensional $O(m+1,1)$-module where the connected component of unity acts trivially and elements $-Id$ and $diag(-1,1,...,1)$ acts as (-1).

Formula (24) defines canonical nondegenerate pairing

$$< \cdot, \cdot >_{S^m} : \Phi^+_{-\lambda-m}(S^m) \otimes \Phi^-_\lambda(S^m) \longrightarrow \varepsilon(1) \tag{32}$$

Further, there is nondegenerate pairing

$$< \cdot, \cdot >_{\mathcal{N}_\lambda} : \quad Sol^+_{C^\infty(\mathcal{N}_\lambda)} \otimes Sol^-_{C^\infty(\mathcal{N}_{-\lambda-m})} \longrightarrow \varepsilon(1); \quad < \varphi, v >_{\mathcal{N}_\lambda} := \int_{\gamma^+_m} \omega_m(\varphi, v)$$

Indeed, the involution $\xi \longmapsto -\xi$ multiplies the form $\omega_m(\varphi, v)$ by $(-1)^{m+2}$ and the cycle γ^+_m by $(-1)^{m+1}$. The involution $diag(-1,1,...,1)$ multiplies by (-1) the orientation of the space, cycle γ^+ and the form $\omega_m(\varphi, v)$. This provides the extra factor $\varepsilon(1)$ in the above formulas. Notice that $\Phi^-_\lambda(S^m) \otimes \varepsilon(1) = \Phi^+_\lambda(S^m)$.

Consider the kernels

$$\bar{K}^-_\lambda(\xi, x) := \frac{| < \xi, x > |^\lambda \cdot sgn(< \xi, x >)}{\Gamma(\frac{\lambda+2}{2})} \qquad \bar{K}^+_\lambda(\xi, x) := \frac{| < \xi, x > |^\lambda}{\Gamma(\frac{\lambda+1}{2})} \tag{33}$$

Let us denote by $\mathcal{M}_\lambda$ the $\mathcal{D}$-module on $\mathbb{R}^{m+2}$ corresponding to the system $\{L_{-\lambda} f = 0, \quad (x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2) f = 0\}$. Then

$$\bar{K}^\pm_\lambda(\xi, x) \in Sol\Big(\mathcal{M}_\lambda \boxtimes \mathcal{N}_\lambda, \mathcal{D}'(\mathbb{R}^{m+2} \times (\mathbb{R}^{m+2})')\Big)$$

$\bar{K}^+_\lambda(\xi, x)$ is an even solution and $\bar{K}^-_\lambda(\xi, x)$ is an odd one. Notice that $\mathcal{M}_\lambda = \tilde{*}\mathcal{M}_{-\lambda-m}$ and $\mathcal{N}_\lambda = \tilde{*}\mathcal{N}_{-\lambda-m}$. So they define integral operators

$$I^+_\lambda : \Phi^+_{-\lambda-m}(S^m) \longrightarrow Sol_{C^\infty}(\mathcal{N}_\lambda)^+; \quad J^-_\lambda : Sol_{C^\infty}(\mathcal{N}_{-\lambda-m})^+ \longrightarrow \Phi^-_\lambda(S^m) \otimes \varepsilon(1) \tag{34}$$

$$(I^+_\lambda f)(\xi) = \int_{\beta_m} \delta(x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2) f(x) \bar{K}^+_\lambda(\xi, x)\sigma_{m+2}(x, dx) \tag{35}$$

$$(J_\lambda \varphi)(\xi) = \frac{1}{2}\int_K \omega_m(\varphi; \bar{K}^-_\lambda(\xi, x)) \tag{36}$$

Theorem 4.8 *a) These operators are intertwiners for the group $O(m+1,1)$ and one has*

$$d(K) \cdot J^-_{-\lambda-m} \circ I^+_\lambda = \frac{\pi^{m+1}}{\Gamma(\frac{-\lambda}{2})\Gamma(\frac{\lambda+m+2}{2})} Id$$

b)For an admissible family Γ_C the operator J^-_λ provides an operator $J^-_{\lambda,C} : \Psi^+_\lambda(\Gamma_C) \longrightarrow \Phi_{-\lambda-m}(S^m)$ which is the inversion of the integral operator $I^+_{\lambda,C} : \Phi_{-\lambda-m}(S^m) \longrightarrow \Psi^+_\lambda(\Gamma_C)$.

Remark. Interchanging odd and even kernels one can similarly define another pair of intertwiners.

The operator J^-_λ is intertwiner thanks to the following reasons.

1. A group element $g \in SO(m+1,1)_0$ sends form $\omega_m(\varphi, v)$ to the form $\omega_m(g \cdot \varphi, g \cdot v)$. Indeed, the form ω_m is a cocyle representing the Green class for the system $\mathcal{N}_\lambda$. This system as well as the volume form in $\mathbb{R}^{m+2}$ is invariant under the action of the group $SO(m+1,1)_0$.

2. A *connected* Lie group acts trivially on the homology.

3. One can show that J^-_λ commutes with $-Id$ and $diag(-1, 1, ..., 1)$.

In the defintion of the inverse operator J^-_λ we can integrate over an m-cycle $\tilde{K} \subset (\mathbb{R}^{m+2})'$ projecting to K. So J^-_λ apriory defined for any smooth function $\varphi(\xi)$. However it commutes with the group action only on the subspace $Sol(\mathcal{N}_\lambda, C^\infty(\mathbb{R}^{m+2}))$. Indeed, g moves the cycle $\tilde{K}$ to a different cycle $g\tilde{K}$ *homologous to the initial one.* To compare the integrals we use the Stokes formula for the form $\omega_m(\varphi; K_\lambda(\xi, x))$. The integrals will be the same only if the form is closed. This happened only if $\varphi(\xi) \in Sol_{C^\infty(\mathcal{N}_\lambda)}$.

The generalized functions (33) has no poles on λ. One has

$$\bar{K}^+_\lambda(\xi, x)|_{\lambda=-(2k+1)} = \frac{(-1)^k k!}{(2k)!} \cdot \delta^{(2k)}(< \xi, x >);$$

$$\bar{K}^-_\lambda(\xi, x)|_{\lambda=-2k} = \frac{(-1)^k (k-1)!}{(2k-1)!} \cdot \delta^{(2k-1)}(< \xi, x >)$$

So theorem (4.4) is a special case of theorem (4.8).

5 The bicategory of D-modules

1. Motivations. How to compute the composition of natural linear maps between solution spaces? A closely related problem is the inversion of a given natural linear map.

Usually the natural kernels are *distributions satisfying holonomic system of differential equations.* This means that the image of homomorphism

$$\check{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2 \longrightarrow \mathcal{D}'(X_1 \times X_2) \tag{37}$$

provided by the kernel $K_{12}(x_1, x_2) \in Hom_{\mathcal{D}}\Big(\check{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{D}'(X_1 \times X_2)\Big)$ is a holonomic $\mathcal{D}$-module. Let us denote it by $\mathcal{K}_{12}$ and by $\check{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2 \xrightarrow{\alpha_{12}} \mathcal{K}_{12}$ the corresponding morphism of $\mathcal{D}$-modules. So (37) is a composition

$$\check{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2 \xrightarrow{\alpha_{12}} \mathcal{K}_{12} \hookrightarrow \mathcal{D}'(X_1 \times X_2)$$

We will keep only the first arrow and call it *a holonomic kernel.*

Example. Suppose that $\mathcal{M}_i = \mathcal{D}_{X_i}$ for $i = 1, 2$. Then $\tilde{\star}\mathcal{D}_{X_1} = \mathcal{D}_{X_1}$ and $\mathcal{D}_{X_1} \times \mathcal{D}_{X_2} = \mathcal{D}_{X_1 \times X_2}$. Morphisms of $\mathcal{D}$-modules $\mathcal{D}_{X_1 \times X_2} \longrightarrow \mathcal{K}$ are defined by their value on the generating section 1 and correspond just to the *sections* of $\mathcal{K}$.

For instance, if $X_1 = X_2 = \mathbb{A}^1$ and $\mathcal{K}_{12}$ is the $\mathcal{D}$-module of delta functions on the diagonal the morphisms above correspond to sections $f(x)\delta^{(k)}(x - y)$.

So holonomic kernel is a finer algebraic version of a holonomic distribution on $X_1 \times X_2$ then the $\mathcal{D}$-module which this distribution satisfies.

2. A bicategory of $\mathcal{D}$-modules. For a definition of (lax) bicategory see, for example, p.200 in [KV].

In this section we work in the derived category. In particulary all morphisms are morphisms in the derived category. So we will write f_* instead of Rf_* etc.

The objects of the bicategory are pairs $(X, \mathcal{M})$ where X is an algebraic variety over a field k (char $k = 0$) and $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$.

A 1-morphism between the 2 objects $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ is a holonomic complex of $\mathcal{D}$-modules $K \in D^b_{hol}(\mathcal{D}_{X \times Y})$ on $X \times Y$ together with a morphism

$$\star\mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\alpha} K$$

A 2-morphism between 1-morphisms

$$\star\mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\alpha_1} \mathcal{K}_1 \quad \text{and} \quad \star\,\mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\alpha_2} \mathcal{K}_2$$

is a morphism $\varphi_{12} : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ making the following diagram commutative:

$$\begin{array}{ccccc} & & \star\mathcal{M}_1 \boxtimes \mathcal{M}_3 & & \\ & \alpha_1 \swarrow & & \searrow \alpha_2 & \\ \mathcal{K}_1 & & \xrightarrow{\varphi_{12}} & & \mathcal{K}_2 \end{array}$$

The composition of 2-morphisms is defined in an obvious way.

It is the composition of 1-morphismes which makes the whole story interesting and relevant to integral geometry. To define it consider objects $(X_i, \mathcal{M}_i)$ where $i = 1, 2, 3$ and 1-morphisms

$$\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \xrightarrow{\alpha_{12}} \mathcal{K}_{12} \qquad \star\,\mathcal{M}_2 \boxtimes \mathcal{M}_3 \xrightarrow{\alpha_{23}} \mathcal{K}_{23} \tag{38}$$

Let $\Delta_2 : X_1 \times X_2 \times X_3 \hookrightarrow X_1 \times X_2 \times X_2 \times X_3$ be the diagonal embedding of X_2 and $\pi_2 : X_1 \times X_2 \times X_3 \to X_1 \times X_3$ be the projection. Set

$$\mathcal{K}_{13} = \mathcal{K}_{12} \circ \mathcal{K}_{23} := \pi_{2*}\Delta_2^!\Big(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}\Big)$$

and define morphism $\star\mathcal{M}_1 \boxtimes \mathcal{M}_3 \xrightarrow{\alpha_{13}} \mathcal{K}_{13}$ as the composition of the morphisms $id \boxtimes G \boxtimes id$ and $\alpha_{12} \boxtimes \alpha_{23}$:

$$\star\mathcal{M}_1 \boxtimes \mathcal{M}_3 \xrightarrow{id \boxtimes G \boxtimes id} \pi_{2*}\Delta_2^!\Big(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \star\mathcal{M}_2 \boxtimes \mathcal{M}_3\Big) \xrightarrow{\alpha_{12} \boxtimes \alpha_{23}} \pi_{2*}\Delta_2^!\Big(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}\Big)$$

Example *The identity 1-morphism* $Id_{\mathcal{M}}$. Let δ_Δ be the $\mathcal{D}$-module of delta functions on the diagonal $\Delta \subset X \times X$. Then for any $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$ there is a canonical morphism

$$I_{\mathcal{M}} : \star\mathcal{M} \boxtimes \mathcal{M}[-d_X] \longrightarrow \delta_\Delta$$

Replacing $\mathcal{M}$ by a locally free resolution we reduce this statement to the case $\mathcal{M} = \mathcal{D}_X$.

The $\mathcal{D}_X - mod - \mathcal{D}_X$ bimodule $\mathcal{D}_X$ is canonically isomorphic to the bimodule $\delta_\Delta \otimes \Omega_X$. The section 1_X of $\mathcal{D}_X$ corresponds to the canonical section $\delta(\Delta)dx$ providing the canonical morphism $I_{\mathcal{D}_X} : \mathcal{D}^\Omega_X \boxtimes \mathcal{D}_X \longrightarrow \delta_\Delta$.

We will say that the 1-morphism α_{23} is weakly inverse to the 1-morphism α_{12} (see (38)) if there is a 2-morphism from the identity 1-morphism $Id_{\mathcal{M}}$ to the composition of 1-morphisms $\alpha_{23} \circ \alpha_{12}$. This means that the following diagram is commutative:

$$\begin{array}{ccc}
\star\mathcal{M}_1 \boxtimes \mathcal{M}_1 & \xrightarrow{id\boxtimes G\boxtimes id} & R\pi_{2*}\Delta_2^!\Big(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \star\mathcal{M}_2 \boxtimes \mathcal{M}_1\Big) \\
\Big\downarrow I_{\mathcal{M}_1} & & \Big\downarrow R\pi_{2*}\Delta_2^!(\alpha_{12}\boxtimes\alpha_{23}) \\
\delta_\Delta & \xrightarrow{\varphi} & R\pi_{2*}\Delta_2^!\Big(\mathcal{K}_{12}\boxtimes\mathcal{K}_{23}\Big)
\end{array}$$

Remark. These definitions make sence for any (not necessarily holonomic) $\mathcal{K}_{ij} \in D^b_{coh}(\mathcal{D}_X)$.

3. An example: the Radon transform on the plane. Set

$$X_1 = \{(x,y)\} = \mathbb{R}^2, \quad X_2 = \{(a,b)\} = \mathbb{R}^2, \quad X_3 = \{(x',y')\} = \mathbb{R}^2$$

and $\mathcal{M}_{X_i} = \mathcal{D}_{X_i}$. Notice that $\star\mathcal{D}_{X_i} \boxtimes \mathcal{D}_{X_{i+1}} = \mathcal{D}_{X_i\times X_{i+1}}[d_{X_i}]$. Set

$$\delta(A) = \delta(y - ax - b) \quad \text{and} \quad \delta(A') = \delta(y' - ax' - b)$$

$$\mathcal{K}_{12} := \mathcal{D}_{X_1\times X_2}\cdot\delta(A), \quad \mathcal{K}_{23} := \mathcal{D}_{X_2\times X_3}\cdot\delta(A')$$

$$\alpha_{12}[-2] : 1_{X_1\times X_2} \longmapsto \delta(A), \quad \alpha_{12}[-2] : 1_{X_2\times X_3} \longmapsto \delta^{(1)}(A')$$

Proposition 5.1 *The formula*

$$\delta(x - x')\delta(y - y') \longmapsto \delta(y - ax - b) \otimes \delta^{(1)}(y' - ax' - b)dadb \tag{39}$$

defines a homomorphism of $\mathcal{D}$-modules $\delta_{\Delta_{13}} \longrightarrow \mathcal{K}_{13}$ and hence a 2-morphism $Id_{\mathcal{D}_{X_1}} => (\alpha_{13}, \mathcal{K}_{13})$.

Proof. We have to show that applying to the right hand side of (39) any differential operator which annihilates the left hand side, we will get exact 2-form in the de Rham complex with respect to (a,b) variables. This follows from the formulas

$$(x - x')\cdot\delta(A)\otimes\delta^{(1)}(A')dadb = d\Big(\delta(A)\otimes\delta(A')(xda + db)\Big)$$

$$(y - y')\cdot\delta(A)\otimes\delta^{(1)}(A')dadb = d\Big(\delta(A)\otimes\delta(A')a(xda + db)\Big)$$

$$(\partial_x + \partial_{x'})\delta(A)\otimes\delta^{(1)}(A')dadb = d\Big(\delta(A)\otimes\delta^{(1)}(A')ada\Big)$$

$$(\partial_y + \partial_{y'})\delta(A)\otimes\delta^{(1)}(A')dadb = d\Big(\delta(A)\otimes\delta^{(1)}(A)da\Big)$$

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