Galois groups, geometry of modular varieties and graphs

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1 Galois groups

1. Galois symmetries. Let X be a regular curve, $\overline{X}$ the corresponding projective curve, and v a tangent vector at a point $x \in \overline{X}$. Then according to Deligne ([D]) one can define the profinite fundamental group $\widehat{\pi}_1(X,v)$ based at the vector v. If X, x and v are defined over a number field F then the group $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ acts by automorphisms of $\widehat{\pi}_1^{(l)}(X,v)$.

If $X = \mathbb{P}^1\backslash\{0,\mu_N,\infty\}$ there is a tangent vector v_1 corresponding to the inverse t^{-1} of the canonical coordinate t on $\mathbb{P}^1\backslash\{0,\mu_N,\infty\}$.

Let us denote by $\widehat{\pi}^{(l)}$ the pro-l-completion of the group $\widehat{\pi}$. In this talk we are going to investigate the map

$$\Phi_N^{(l)}: \quad G_{\mathbb{Q}} \longrightarrow Aut\widehat{\pi}_1^{(l)}(\mathbb{P}^1\backslash\{0,\mu_N,\infty\},v_1) \tag{1}$$

When $N=1$ this map has been studied by Grothendieck [Gr], Deligne [D], Ihara [Ih] , Drinfeld [D], ... (see [Ih]), but for $N>1$ it was not investigated.

Let $\pi_1^{(l)} := \quad \widehat{\pi}_1^{(l)}(\mathbb{P}^1\backslash\{0,\mu_N,\infty\},v_1)$. Denote by $\widehat{\pi}_1^{(l)}(m)$ the lower central series for the group $\widehat{\pi}_1^{(l)}$. Then

$$\widehat{\pi}_1^{(l)}/\widehat{\pi}_1^{(l)}(m) \quad \text{is an } l\text{-adic Lie group}$$

Taking its Lie algebra and making the projective limit over m we get a pronilpotent Lie algebra over $\mathbb{Q}_l$:

$$\mathbb{L}_N^{(l)} := \quad \lim_{\longleftarrow} Lie\Big(\frac{\widehat{\pi}_1^{(l)}}{\widehat{\pi}_1^{(l)}(m)}\Big)$$

Let $\mathbb{Q}(\zeta_n)$ be the field generated by n-th roots of unity. Set $\mathbb{Q}(\zeta_{l^\infty}) := \cup\mathbb{Q}(\zeta_{l^n})$. We restrict map (1) to the Galois group $G_{\mathbb{Q}(\zeta_{l^\infty})}$. Passing to Lie algebras we get a homomorphism

$$\phi_N^{(l)} : G_{\mathbb{Q}(\zeta_{l^\infty})} \longrightarrow \mathrm{Aut}(\mathbb{L}_N^{(l)})$$

We would like to take the Lie algebra of the image of this map. One has to proceed as follows. Let $\mathbb{L}_N^{(l)}(m)$ be the lower central series for the Lie algebra $\mathbb{L}_N^{(l)}$. There are homomorphisms to finite dimensional l-adic Lie groups

$$\phi_{N;m}^{(l)} : G_{\mathbb{Q}(\zeta_{l^\infty})} \longrightarrow \mathrm{Aut}\Big(\mathrm{L}_N^{(l)}/\mathbb{L}_N^{(l)}(m)\Big)$$

The main hero of this story is the pronilpotent Lie algebra

$$\mathcal{G}_N^{(l)} := \quad \lim_{\leftarrow} Lie\Big(Im\phi_{N;m}^{(l)}\Big)$$

Therefore

$$\mathcal{G}_N^{(l)} \hookrightarrow \mathrm{Der}\mathbb{L}_N^{(l)}$$

2. The weight and depth filtration on $\mathbb{L}_N^{(l)}$. There are two increasing filtrations by ideals on the Lie algebra $\mathbb{L}_N^{(l)}$, indexed by negative integers.

The weight filtration $F_\bullet^W$. It coincides with the lower central series for $\mathbb{L}_N^{(l)}$:

$$\mathbb{L}_N^{(l)} = F_{-1}^W\mathbb{L}_N^{(l)}; \quad F_{-n-1}^W\mathbb{L}_N^{(l)} := [F_{-n}^W\mathbb{L}_N^{(l)}, \mathbb{L}_N^{(l)}]$$

The depth filtration $F_\bullet^D$. The natural inclusion

$$\mathbb{P}^1\backslash\{0, \mu_N, \infty\} \hookrightarrow \mathbb{P}^1\backslash\{0, \infty\}$$

provides a morphism of the corresponding fundamental Lie algebras

$$p : \mathbb{L}_N^{(l)} \longrightarrow \mathbb{L}^{(l)}(\mathbb{P}^1\backslash\{0,\infty\}) = \mathbb{Q}_l(1)$$

Let $\mathcal{I}_N$ be the kernel of this projection Its powers give the depth filtration:

$$F_0^D\mathbb{L}_N^{(l)} = \mathbb{L}_N^{(l)}, \quad F_{-1}^D\mathbb{L}_N^{(l)} = \mathcal{I}_N, \quad F_{-n-1}^D\mathbb{L}_N^{(l)} = [\mathcal{I}_N, F_{-n}^D\mathbb{L}_N^{(l)}]$$

The weight filtration can be defined on the Lie algebra corresponding to the pronilpotent completion of the fundamental group of an arbitrary algebraic variety. The weights on $\mathbb{L}_N^{(l)}$ are obtained by dividing by 2 the usual weights. The weight filtration admits a splitting, i.e. it is defined by a grading.

The depth filtration is more specific.

These filtrations induce two filtrations on the Lie algebra $\mathrm{Der}\mathbb{L}_N^{(l)}$ and hence on the Lie algebra $\mathcal{G}_N^{(l)}$. Taking the associated graded we get a Lie algebra $\mathrm{Gr}_{\bullet,\bullet}^{D,W}\mathcal{G}_N^{(l)}$ bigraded by the weight $-w$ and depth $-m$.

Since the weight filtration admits a splitting, i.e. is defined by a grading, and both the depth filtration and the subspace $\mathcal{G}_N^{(l)} \subset \mathrm{Der}\mathbb{L}_N^{(l)}$ are compatible with this grading, one has

$$\mathrm{Gr}_{\bullet,\bullet}\mathcal{G}_N^{(l)} \subset \mathrm{Gr}_{\bullet,\bullet}\mathrm{Der}\mathbb{L}_N^{(l)}$$

3. A mysterious relation. Let $\Gamma_1(m;N)$ is the subgroup of $GL_m(\mathbb{Z})$ stabilizing the vector $(1,0,...,0)$ mod N. Denote by V_m the standard m-dimensional representation of GL_m. The key point of this talk (as well as the papers [G2]-[G5]) is that

the structure of the bigraded Lie algebra $\oplus_{w\geq m\geq 1}\mathrm{Gr}_{-m,-w}\mathcal{G}_N^{(l)}$

is related to the

geometry of the local system with fiber $S^{w-m}\mathrm{V}_m$ *over the modular variety*

$$\Gamma_1(m;N)\backslash GL_m(\mathbb{R})/SO_m\cdot\mathbb{R}^*$$

The Lie algebra $\mathbb{L}_N^{(l)}$ is the l-adic realization of a certain pro-Lie algebra in the category of mixed Tate motives over the scheme $S_N := \mathrm{Spec}\mathbb{Z}[\zeta_N][\frac{1}{N}]$ which will be denoted $\mathbb{L}_N^{\mathcal{M}}$. It is the "motivic fundamental group" (see [D]) of $\mathbb{P}^1\backslash\{0,\mu_N,\infty\}$. The Hodge realization of the motive $\mathbb{L}_N^{\mathcal{M}}$ is described by multiple polylogarithms at N-th roots of unity (see [G5]).

4. Examples of this relationship. For any $m\geq 1$ there is the depth $\geq -m$ quotient Lie algebra

$$\mathrm{Gr}_{\geq -m,\bullet}\mathcal{G}_N^{(l)} := \frac{\mathrm{Gr}_{\bullet,\bullet}\mathcal{G}_N^{(l)}}{\mathrm{Gr}_{<-m,\bullet}\mathcal{G}_N^{(l)}}$$

It is a nilpotent graded Lie algebra of the nilpotence class m. In particular the depth -1 quotient Lie algebra is abelian.

Strangely enough it is more natural to describe the structure of a certain bigraded Lie algebra $\tilde{\mathcal{G}}_N^{(l)}$ which sits in the exact sequence

$$0 \longrightarrow \mathcal{G}_N^{(l)} \longrightarrow \tilde{\mathcal{G}}_N^{(l)} \longrightarrow \mathbb{Q}_l(1) \longrightarrow 0$$

where $\mathbb{Q}_l(1)$ is a one dimensional abelian quotient of $\tilde{\mathcal{G}}_N^{(l)}$ which is of weight -1 and depth -1.

I do not know any motivic or Galois-theoretic meaning of this extension, also the added component $\mathbb{Q}_l(1)$ somehow corresponds to $\zeta(1)$, whatever it means.

0). *Depth* $=-1$, $N=1$. The situation in this case is completely known thanks to the efforts of Soulé [So], Deligne [D], and Ihara [Ih2]. Denote by $\mathcal{G}^{(l)}$ the Lie algebra $\mathcal{G}_1^{(l)}$.

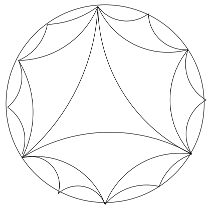
Theorem 1.1

$$\mathrm{Gr}_{-1,-w}\tilde{\mathcal{G}}^{(l)} = \begin{cases} 0 & w \quad \textit{even}, \quad \textit{or } w=1 \\ \mathbb{Q}_l & \quad \textit{odd} \end{cases} \tag{2}$$

1). *Depth* $=-2$, $N=1$. The structure of the depth ≥ -2 quotient of the Lie algebra $\mathrm{Gr}_{\bullet,\bullet}\tilde{\mathcal{G}}^{(l)}$ is completely described by the commutator map

$$[,]: \quad \Lambda^2\mathrm{Gr}_{-1,\bullet}\tilde{\mathcal{G}}^{(l)} \longrightarrow \mathrm{Gr}_{-2,\bullet}\mathcal{G}^{(l)} \tag{3}$$

Construction of the dual to the complex (3). Consider the classical modular triangulation of the hyperbolic plane:



The group $GL_2(\mathbb{Z})$ acts on the hyperbolic plane $\mathcal{H}^2$ preserving the modular picture. Let

$$M^*_{(2)} := \quad M^1_{(2)} \longrightarrow M^2_{(2)} \tag{4}$$

be the chain complex of the modular triangulation. Here the group $M^1_{(2)}$ is generated by the triangles, and the group $M^2_{(2)}$ by the geodesics. It is a complex of $GL_2(\mathbb{Z})$-modules.

Theorem 1.2 *The depth w part of the dual to the complex (3) is* **canonically** *isomorphic to the complex*

$$M^*_{(2)} \otimes_{GL_2(\mathbb{Z})} S^{w-2}\mathrm{V}_2 \tag{5}$$

The complex (5) computes the cohomology of the group $GL_2(\mathbb{Z})$ with coefficients in $S^{w-2}\mathrm{V}_2$. Since these cohomology groups are well known, using theorems 1.1, 1.2 we immediately get the following result (compare it with [Ih3]):

Corollary 1.3

$$\dim\mathrm{Gr}_{-2,-w}\tilde{\mathcal{G}}^{(l)} = \begin{cases} 0 & w: \quad \textit{odd} \\ \left[\frac{w-2}{6}\right] & w: \quad \textit{even} \end{cases} \tag{6}$$

2). *$N=p$ is a prime; $w=m$.*
2i) $w=m=-1$. In this case it is not dificult to show that

$$\mathrm{Gr}_{-1,-1}\mathcal{G}^{(l)}_p = (\text{the group of the cyclotomic units in } \mathbb{Z}[\zeta_p][\tfrac{1}{p}]) \otimes_{\mathbb{Z}} \mathbb{Q}_l$$

2ii) $w=m=-2$. Let $Y_1(p) := \Gamma_1(p)\backslash\mathcal{H}^2$ be an open part of the modular curve. Projecting the modular triangulation of the hyperbolic plane onto $Y_1(p)$ we get the modular triangulation of $Y_1(p)$. The complex involution acts on the modular curve preserving the triangulation. Consider the following complex

$$\Big(\text{the chain complex of the modular triangulation of } Y_1(p)\Big)^+ \qquad (7)$$

Here $+$ means the invariants of the action of the complex involution.

Theorem 1.4 *The dual to the complex*

$$[,]: \quad \Lambda^2\mathrm{Gr}_{-1,-1}\tilde{\mathcal{G}}_p^{(l)} \longrightarrow \mathrm{Gr}_{-2,-2}\tilde{\mathcal{G}}_p^{(l)}$$

is naturally **isomorphic** *to the complex (7).*

In particular

$$\mathbb{Q}_l[\text{triangles of the modular triangulation of } Y_1(p)]^+ = \Big(\mathrm{Gr}_{-2,-2}\mathcal{G}_p^{(l)}\Big)^\vee$$

Remark. In [G2], [G5] we were dealing with the depth -2, weight -2 quotient of $\mathcal{G}_p^{(l)}$, not $\tilde{\mathcal{G}}_p^{(l)}$, and so we get only quasiisomorphism with a complex similar to (7).

2 General picture: the modular complexes and Galois symmetries of $\widehat{\pi}_1^{(l)}(\mathbb{P}^1\backslash\{0,\mu_N,\infty\},v_1)$

Let L_m be a rank m lattice. The modular complex $M^\bullet(L_m) = M^\bullet_{(m)}$ is a complex of $GL_m(\mathbb{Z})$-modules

$$M^1_{(m)} \xrightarrow{\partial} M^2_{(m)} \xrightarrow{\partial} \dots \xrightarrow{\partial} M^m_{(m)}$$

If $m=2$ it coincides with complex (4). In general it is defined as follows.

i) *The group* $M^1_{(m)}$.

Definition 2.1 *An affine basis of a lattice L_m is an unordered $(m+1)$-tuple of vectors $v_1,...,v_{m+1}$ of the lattice such that $v_1+...+v_{m+1}=0$ and $v_1,...,v_m$ is a basis.*

The set $\mathcal{A}_m$ of affine basises $(v_1,...,v_{m+1})$ of the lattice L_m is a principal homogeneous space of $GL_m(\mathbb{Z})$.

The abelian group $M^1(L_m) = M^1_{(m)}$ is generated by the elements $< e_1,...,e_{m+1} >$ where $e_1,...,e_{m+1}$ runthrough the set of all affine basises. So $M^l_{(m)} := \mathbb{Z}[\mathcal{A}_m]/R^l_{(m)}$ where the subgroup of relations $R^l_{(m)}$ will be defined below.

To list the relations we need another set of the generators:

$$< u_1 : ... : u_{m+1} >:=< u_2 - u_1, u_3 - u_2, ..., u_1 - u_{m+1} >$$

Relations. Let $\Sigma_{k,m-k}$ be the set of all $(k, m-k)$ shuffles of the ordered sets $\{1, ..., k\}$ and $\{k+1, ..., m\}$. For any $1 \leq k \leq m$ one has

$$\sum_{\sigma \in \Sigma_{k,m-k}} < v_{\sigma(1)}, ..., v_{\sigma(m)}, v_{m+1} > \quad = \quad 0 \tag{8}$$

$$\sum_{\sigma \in \Sigma_{k,m-k}} < u_{\sigma(1)} : ... : u_{\sigma(m)} : u_{m+1}] \quad = \quad 0 \tag{9}$$

ii) *The group* $M^k_{(m)}$. Let

$$L_m = L^1 \oplus ... \oplus L^k \tag{10}$$

be a decomposition of the lattice L_m into a direct sum of sublattices. Then $M^k_{(m)}$ is the product of the groups

$$M^1(L^1) \wedge ... \wedge M^1(L^k)$$

over all unordered decompositions (10). In other words it is generated by the elements $< A_1 > \wedge ... \wedge < A_k >$ where A_i is an affine basis of the sublattice L_i.

Let us set

$$[v_1, ..., v_k] :=< v_1, ..., v_k, v_{k+1} >, \quad v_1 + ... + v_k + v_{k+1} = 0$$

We define a differential

$$\partial : M^1_{(m)} \longrightarrow M^2_{(m)} \tag{11}$$

on the generators by the formula

$$\partial : \quad < v_1, ..., v_{m+1} > \quad \longmapsto \quad -\text{Cycle}_{m+1}\Big(\sum_{k=1}^{m-1} [v_1, ..., v_k] \wedge [v_{k+1}, ..., v_m]\Big)$$

where the indices are modulo $m+1$ and

$$\text{Cycle}_{m+1}\Big(f(v_1, ..., v_m)\Big) := \quad \sum_{i=1}^{m+1} f(v_i, ..., v_{i+m})$$

We extend the differential to the whole complex $M^\bullet_{(m)}$ using the Leibniz rule:

$$\partial([A_1] \wedge [A_2] \wedge ...) := \quad \partial([A_1]) \wedge [A_2] \wedge ... \quad - \quad [A_1] \wedge \partial([A_2]) \wedge ... \quad +$$

Theorem 2.2 *There exists a canonical homomorphism of complexes*

$$M^*_{(m)} \otimes_{GL_m(\mathbb{Z})} \Big(\mathbb{Z}[\Gamma_1(m;N)\backslash GL_m(\mathbb{Z})] \otimes_{\mathbb{Z}} S^{w-m}\mathrm{V}_m\Big) \longrightarrow \tag{12}$$
$$\Big(\Lambda^*\mathrm{Gr}_{\bullet,\bullet}\tilde{\mathcal{G}}_N^{(l)^\vee}\Big)_{depth=-m,weight=-w}$$

Map (12) was defined on generators in [G5]. It was shown in [G5] that the second shuffle relations go to zero. What was left is a proof of the fact that the first shuffle relations go to zero. See [G4].

Conjecture 2.3 *Let $N = 1$. Then homomorphism (12) is an isomorphism.*

Conjecture 2.4 *Let $N = p$ be a prime and $w = m$. Then homomorphism (12) is an isomorphism.*

Theorem 2.5 *Conjectures 2.3 and 2.4 are true for $m = 2, 3$.*

Remark. We almost proved they are also true for $m = 4$.

In general homomorphism (12) has a nontrivial kernel which contains the so-called distribution relations (see [G5]) and probably can be also described explicitely. So it should be close to be a quasiisomorphism.

The relation with Galois groups insures that the modular complex is an important mathematical object. In the next section we develope some geometry staying behind it and show how it is related with the rank m symmetric space.

3 Complex of colored forests and its geometric realization in the symmetric space

1. Voronoi's cell decomposition of the symmetric space for $GL_m(\mathbb{R})$. Let

$$\mathbb{H}_m \quad := \quad GL_m(\mathbb{R})/SO(m)\cdot\mathbb{R}^* \quad = \frac{> 0 \text{ definite quadratic forms on } V_m^*}{\mathbb{R}^*_+}$$

Recall that $L_m \subset V_m$ is a lattice. Any vector $v \in V_m$ defines a nonnegative definite quadratic form $\varphi(v) :=< v, \cdot >^2$ on V_m^*. Take the convex hull $\mathcal{C}(L_m)$ of the forms $\varphi(l)$ when l runs through all non zero primitive vectors of the lattice L_m. It has a structure of an infinite polyhedra. Let $GL(L_m) \subset GL(V_m^*)$ be the subgroup preserving the lattice L_m. The projection of $\mathcal{C}(L_m)$ into the closure of $\mathbb{H}_m$ defines a $GL(L_m)$-invariant polyhedral decomposition of $\mathbb{H}_m$.

Example. In the case $m = 2$ we get the modular triangulation of the hyperbolic plane.

Denote by $V_\bullet^{(m)}$ the chain complex of the Voronoi decomposition. It is called the Voronoi complex of rank m.

2. The relaxed modular complex. Consider a version $\widehat{M}_{(m)}^\bullet$ of the modular complex, called the relaxed modular complex, where the group $\widehat{M}_{(m)}^1$ is defined using the same generators $[v_1, ..., v_m]$ satisfying only the dihedral symmetry relations

$$< v_2, ..., v_{m+1}, v_1 >=< v_1, ..., v_{m+1} >= (-1)^{m+1} < v_{m+1}, ..., v_1 >$$

and the first shuffle relations (8). The other groups are defined in a similar way. The differentials are the same as before.

3. The geometric realization. Let us define the join $*$ as follows. Let $v_1, ..., v_{n_1}$ and $v_{n_1+1}, ..., v_{n_1+n_2}$ be two sets of vectors of the lattice such that the lattices generated by these sets of vectors has zero intersection. Then we set

$$\varphi(v_1, ..., v_{n_1}) * \varphi(v_{n_1+1}, ..., v_{n_1+n_2}) := \quad \varphi(v_1, ..., v_{n_1+n_2})$$

and extend the join by linearity.

Let us make a homological complex $\widehat{M}_\bullet^{(m)}$ out of the complex $\widehat{M}_{(m)}^\bullet$ by setting

$$\widehat{M}_i^{(m)} := \widehat{M}_{(m)}^{2m-1-i}$$

Theorem-Construction 3.1 *There exists a canonical morphism of complexes*

$$\widehat{\psi}_\bullet^{(m)} : \widehat{M}_\bullet^{(m)} \longrightarrow V_\bullet^{(m)}$$

such that:

$$\widehat{\psi}_\bullet^{(m)}\Big([A_1] \wedge ... \wedge [A_k]\Big) := \quad \widehat{\psi}_\bullet^{(m)}([A_1]) * ... * \widehat{\psi}_\bullet^{(m)}([A_k]) \qquad (13)$$

Here $[A_1] := [v_1, ..., v_{l_1}]$ and so on.

In particular one has $\widehat{\psi}_{m-1}^{(m)}\Big([v_1] \wedge ... \wedge [v_m]\Big) = \varphi(v_1, ..., v_m)$.

To define such a morphism $\widehat{\psi}_\bullet^{(m)}$ one needs only to define $\widehat{\psi}_{2m-2}^{(m)}([v_1, ..., v_m])$ for vectors $v_1, ..., v_m$ forming a basis of the lattice L_m in such a way that the dihedral and the first shuffle relations go to zero and

$$d\widehat{\psi}_{2m-2}^{(m)}([v_1, ..., v_m]) = \widehat{\psi}_{2m-3}^{(m)}(\partial[v_1, ..., v_m])$$

where the right hand side is computed by (13) and the formula for ∂.

4. The map $\widehat{\psi}_{2m-2}^{(m)}$. A plane tree is a tree without self intersections located on the plane.

Let $V(E)$ be the $\mathbb{R}$-vector space generated by the edges of a tree. By definition an orientation of the tree is a choice of the connected component of $det(V(E))\backslash 0$. So orientations of a tree T form a $\mathbb{Z}/2\mathbb{Z}$ torsor $\mathcal{O}_T$.

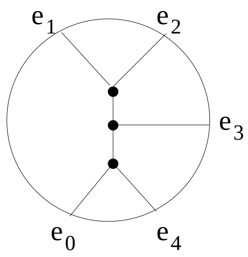
Let v be a vertex of a tree T, and $\mathcal{O}_v$ the $\mathbb{Z}/2\mathbb{Z}$ torsor of orientations of the link of v. It is easy to see that if T is a 3-valent tree then $\mathcal{O}_T = \otimes \mathcal{O}_v$ where the tensor product is over all vertices of T. Therefore a plane 3-valent tree has a canonical orientation. Indeed, the orientation of the plane provides orientations of links of each of the vertices.

The edges of a tree consist of legs (i.e. external edges) and internal edges. In other words a leg is an edge of the tree such that one or both of its vertices have valency one. A tree may have no internal vertices; for example we allow a tree to have only one internal edge. We do not allow however vertices of valency two.

Choose once forever a lattice L_m. A colored tree is a tree whose legs are in a bijective correspondence with the elements of an affine basis of the lattice L_m.

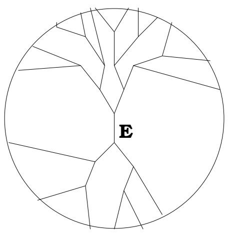
In particular a colored 3-valent tree has $2m - 1$ edges.

Examples. We visualize a colored tree as follows:



The vectors $e_0, ..., e_m$ of an affine basis are located cyclically on an oriented circle and the legs of the tree end on the circle and labelled by $e_0, ..., e_m$. (The circle itself is not a part of the graph).

Construction. Each edge E of the tree T provides a vector $f_E \in L_m$ defined up to a sign. Namely, the edge E determines two trees rooted at E, see the picture



The union of the incoming (i.e. different from E) legs of these rooted trees coincides with the set of all legs of the initial tree. Take the sum of all the vectors e_i corresponding to the incoming legs of one of these trees. Denote it by f_E. If we choose the second rooted tree the sum will change the sign. So the degenerate quadratic form $\varphi(f_E)$ is well defined.

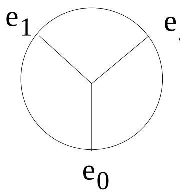
Let us denote by $\varphi(v_1, ..., v_k)$ the convex hull of the forms $\varphi(v_1)$, ..., $\varphi(v_k)$ in the space of quadratic forms. Set

$$\widehat{\psi}^{(m)}_{2m-2}(< e_0, e_1, ..., e_m >) :=$$

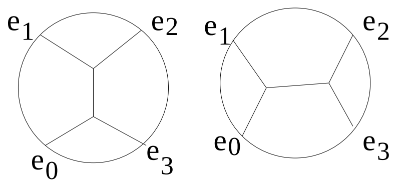
$$\sum_{\text{plane 3-valent trees}} \mathrm{sgn}(E_1 \wedge ... \wedge E_{2m-1}) \cdot \varphi(f_{E_1}, ..., f_{E_{2m-1}}) \qquad (14)$$

Here the sum is over all plane 3-valent trees colored by $e_0, ..., e_m$ and the sign is taken with respect to the canonical orientation of the plane 3-valent tree.

Examples. a) For $m = 2$ there is one plane 3-valent tree colored by e_0, e_1, e_2, so we get the triangle $\varphi(e_0, e_1, e_2)$.



m=2



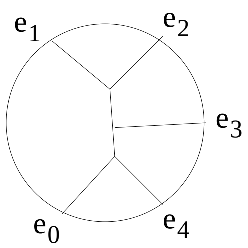
m=3

b) Let $f_{ij} := e_i + e_j$. For $m = 3$ there are two plane 3-valent trees colored by e_0, e_1, e_2, e_3, see the picture, so the chain is

$$\varphi(e_0, e_1, e_2, e_3, f_{01}) - \varphi(e_0, e_1, e_2, e_3, f_{12})$$

c) For $m = 4$ there are five plane 3-valent trees colored by e_0, e_1, e_2, e_3, e_4. We showed one of them on the picture below; the rest are obtained by rotations of the tree. Therefore the chain is

$$\mathrm{Cycle}_5 \varphi(e_0, e_1, e_2, e_3, e_4, f_{12}, f_{34})$$

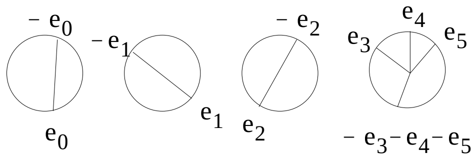


m=4

5. The complex of colored forests. *Terminology.* A forest is a graph without loops. Any forest T is a union if its connected components, i.e. trees: $T = T_1 \cup ... \cup T_k$.

A colored forest is a forest T equipped with the following data. Let $T = T_1 \cup ... \cup T_k$ be the decomposition on trees, and $L^{(1)} \oplus ... \oplus L^{(k)} = L_m$ be a decomposition into a direct sum of k nonzero sublattices such that the number of legs of the tree T_i is equal to $\mathrm{rk} L^{(i)} + 1$. Choose an affine basis in each of the sublattices $L^{(i)}$ and label the legs of the tree T_i by the vectors of this basis (i.e. choose a bijection between the legs and the vectors of the basis). Then we will say that the forest T is colored (by the vectors of the affine basises of the lattices $L^{(i)}$).

Here is an example of a colored forest:



We define the $\mathbb{Z}/2\mathbb{Z}$ torsor of orientations of a forest T by $\mathcal{O}_T = \otimes_{i=1}^k \mathcal{O}_{T_i}$.

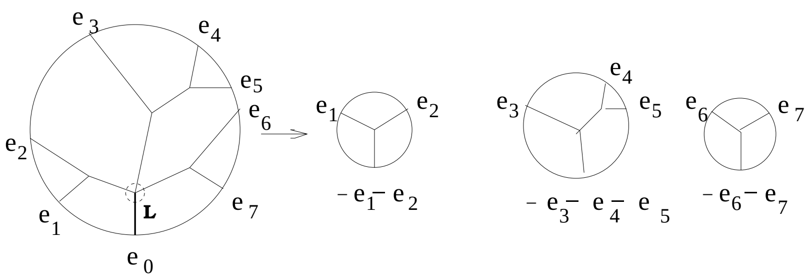
Let $F^i_{(m)}$ be the abelian group generated by the isomorphism classes of pairs (T, Or_T) where T is a colored forest with $2m - i$ edges, and Or_T is an orientation of the forest. The only relation is this: if we change the orientation of T, the generator is multiplied by -1.

Example. The group $F^1_{(m)}$ is generated by colored 3-valent trees with $m + 1$ legs.

Let us define a differential $d : F^i_{(m)} \longrightarrow F^{i+1}_{(m)}$. One has $d = d_I + d_L$.

The map d_I is defined as follows. Let (T, Or_T) be a generator of $F^i_{(m)}$ and E an *internal* edge of T. Let T/E be the tree obtained by contraction of the edge E. So it has one less edge, and one less vertex. The orientation Or_T of the graph T induces a natural orientation $\mathrm{Or}_{T/E}$ of the graph T/E. Namely, if Or_T is represented by $E \wedge E_1 \wedge E_2 \wedge ...$ then $\mathrm{Or}_{T/E}$ is given by $E_1 \wedge E_2 \wedge ...$. Notice that contructing an internal edge we do not touch the coloring. So d_I is essentially the differential used by Boardman and Kontsevich.

Let us define the map d_L. Let L be a leg of T. Let us remove this leg together with a little neighborhood of its vertices. If both of the vertices of L has valency 1 this means that L is a connected component of T, so we just removed this connected component. Otherwise one of the vertices has the valency $v \geq 3$, and the connected component of the tree containing the leg L is replaced by $v - 1$ connected trees. Namely, each of the internal edges sharing the vertex with L produces a new tree. The coloring of the tree T provides a natural coloring of the new tree, see the picture.



More precisely, the coloring of the legs of T different from L remains unchanged; each of the new trees has one new leg, and the vector attached to it is the minus sum of the vectors atached to the rest of the legs of this tree.

By definition the component of the differential corresponding to a leg L which itself forms a tree is zero.

It is easy to see that we get a complex, denoted $F^{\bullet}_{(m)}$. We call it the complex of colored forests attached to a lattice L_m. It is different from the complex of trees/graphs considered by Boardman and Kontsevich.

6. A geometric realization of the complex of colored forests. Let us make a homological complex $F^{(m)}_{\bullet}$ out of $F^{\bullet}_{(m)}$ by setting $F^{(m)}_i := F^{2m-i-1}_{(m)}$.

Theorem-Construction 3.2 *There exists a canonical injective homomorphism of complexes*

$$\tau^{(m)}_{\bullet} : F^{(m)}_{\bullet} \longrightarrow V^{(m)}_{\bullet}$$

Proof. Let (T, Or_T) be a generator of $\mathrm{F}^{(m)}_i$.

Let $E_1, ..., E_{2m-i}$ be the set of all edges of T. Let $\mathrm{sgn}(E_1 \wedge ... \wedge E_{2m-i}) = +1$ if $\mathrm{Or}_T = E_1 \wedge ... \wedge E_{2m-i}$, and -1 otherwise. Set

$$\tau^{(m)}_i : (T, \mathrm{Or}_T) \longmapsto \mathrm{sgn}(E_1 \wedge ... \wedge E_{2m-i}) \cdot \varphi(f_{E_1}, ..., f_{E_{2m-i}}) \in V^{(m)}_i$$

We get a morphism of complexes. Indeed, the cell $\varphi(f_{E_1}, ..., f_{E_{2m-i}})$ is a simplex, so its boundary is a union of simplices corresponding to the edges of the tree. The contribution of an *internal* edge E is precisely the cell corresponding to $(T/E, \mathrm{Or}_{T/E})$. Similarly the contribution to the boundary of a leg L of the tree T is given by the d_i component of the differential.

Theorem 3.3 *The complex of colored forests is a resolution of the relaxed modular complex.*

Proof. The complex of the colored forests has a filtration by the number of connected components of the forest. The E_1-term of the corresponding spectral sequence is a sum (over affine basises) of the tree complexes. By the Ginzburg-Kapranov theorem [GK] the tree complex with $m + 1$ legs is a resolution of the space of m-ar operations in the cyclic Lie cooperad $CLie^*_{m+1}$. It remains to notice that

$$M^1(L_m) = \oplus_{\text{affine} \quad \text{basises}} CLie^*_{m+1}$$

In particular the relaxed modular complex is a subcomplex of colored forests. Combining this with the geometric realization given above we get the map $\widehat{\psi}^{(m)}_\bullet$.

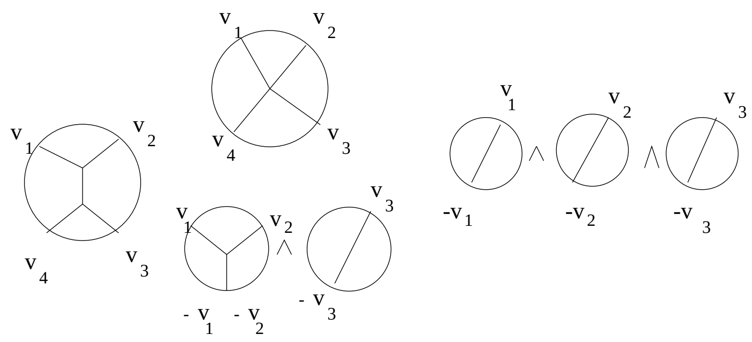
Example. The rank 3 complex of colored forests $F^{(3)}_\bullet$ is isomorphic to the 4-sceleton of the Voronoi complex for GL_3. Indeed, the list of the cells of the Voronoi decomposition of $\mathbb{H}_3$ considered up to $GL_3(\mathbb{Z})$ equivalence is the following (see [G3]):

$$4-\text{cell}: \qquad \varphi(v_1, v_2, v_3, v_4, f_{12})$$

$$3-\text{cells}: \qquad \varphi(v_1, v_2, v_3, v_4), \quad \varphi(v_1, v_2, f_{12}, v_3)$$

$$2-\text{cell}: \qquad \varphi(v_1, v_2, v_3),$$

They corresponds presisely to the colored forests listed on the picture.



4-cells 3-cells 2-cells

A similar but easier investigation shows that $F^{(2)}_\bullet$ is isomorphic to the Voronoi complex for GL_2.

7. Modular complexes and the Voronoi complexes. Here is one of the essential ingrediets of the proof of theorem 2.5. Its proof useses the machinery developed in this section.

Theorem 3.4 *a) There exist isomorphisms of complexes*

$$M^{(2)}_\bullet \longrightarrow V^{(2)}_\bullet, \qquad M^{(3)}_\bullet \longrightarrow \tau_{[4,2]}(V^{(3)}_\bullet)$$

b) There exist a quasiisomorphism of complexes

$$M^{(4)}_\bullet \longrightarrow \tau_{[6,3]}(V^{(4)}_\bullet)$$

The part a) was proved in [G3]. For the part b) see [G4].

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