Quantum symmetry from quantum geometry of moduli spaces

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Based on the of joint works with Linhui Shen,
continuing series of works with Vladimir Fock.

From Galois and Einstein, the idea of symmetry plays key role in Mathematics and Physics.

After works of Yang and Baxter, the idea of quantum symmetry emerged in 70-80's in the works of Leningrad school: Faddeev, Sklyanin, Takhtajan, Reshetikhin ... Precise form discovered by Drinfeld and Jimbo.

What is quantum symmetry?

Symmetry: A (Lie) group G acts on a set/space X: $\mathrm{G} \times X \to X$.

Quasiclassical variant: Groups, spaces and maps become Poisson. So a Poisson Lie group (Drinfeld 1982) is a Lie group G with a Poisson structure, such that the inverse $\mathrm{G} \to \mathrm{G}$ and the composition $\mathrm{G} \times \mathrm{G} \to \mathrm{G}$ are Poisson maps. New aspect: duality of Poisson Lie groups $\mathrm{G} \longleftrightarrow \mathrm{G}^*$.

Quantum groups (Drinfeld, Jimbo $\sim$ 1986). According to quantum mechanics, the algebra of functions on a Poisson space X should be quantized to a noncommutative algebra $\mathcal{O}_q(X)$. A Poisson action $\mathrm{G} \times X \to X$ should be quantized to a map $\mathcal{O}_q(X) \to \mathcal{O}_q(X) \otimes \mathcal{O}_q(\mathrm{G})$. So if G is a Poisson Lie group, the algebra $\mathcal{O}_q(\mathrm{G})$ should become a Hopf algebra. E.g. quantum universal enveloping algebra: $U_q(\mathfrak{g}) = \mathcal{O}_q(\mathrm{G}^*)$.

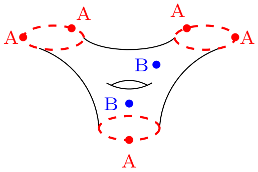
Example: $U_q(\mathfrak{sl}_2)$: the algebra with generators $\mathbf{E}, \mathbf{F}, \mathbf{K}^{\pm 1}$, relations
$[\mathbf{E}, \mathbf{F}] = (q^{-1} - q)(\mathbf{K} - \mathbf{K}^{-1})$, $\mathbf{KEK}^{-1} = q^2\mathbf{E}$, $\mathbf{KFK}^{-1} = q^{-2}\mathbf{F}$.
This algebra appeared first in the work of Kulish-Reshetikhin in 1982.
An explicit coproduct, discovered later, makes it into a Hopf algebra.

It is highly non-trivial to define even simplest Poisson/quantum groups:

Problem: To define what is quantum symmetry we first break the symmetry. For example, to define $U_q(\mathfrak{sl}_2)$ we have to choose a basis in $\mathbb{C}^2$, to introduce the standard linear basis (e, f, h) in $\mathfrak{sl}_2$.

Goal of the lecture: to explain that quantum symmetry arises naturally from quantum geometry of moduli spaces of $\mathrm{G}-$local systems on decorated surfaces, equipped with some extra boundary data. Key role in quantization plays cluster Poisson structure of these moduli spaces.

INPUT:
G: a split semi-simple (adjoint) algebraic group $/\mathbb{Q}$ - say PGL_m.
$\mathbb{S}$: a *decorated surface* - an oriented surface with boundary, with punctures and special boundary points - corners - modulo isotopy.



OUTPUT:
Moduli space $\mathcal{P}_{\mathrm{G},\mathbb{S}}$ of G-local systems on $\mathbb{S}$ with boundary data:
(i) A flag B near each puncture, invariant under local monodromy.

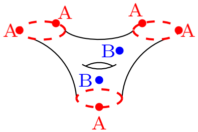
(ii) A decorated flag A at each corner. Pair of flags on each boundary interval - generic.

Flag variety $\mathcal{B} = \mathrm{G}/\mathrm{B}$ (B is a Borel subgroup in G).

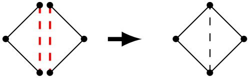
Decorated flag variety $\mathcal{A} := \mathrm{G}/\mathrm{U}$ (U is a max. unipotent in G).

Example. $\mathrm{G} = \mathrm{SL}_2$. So $\mathcal{B} = \mathrm{P}^1$ and $\mathcal{A} = \mathrm{A}^2 - \{0, 0\}$.

Then $\mathcal{P}_{\mathrm{SL}_2, \mathbb{S}}$ parametrises 2d flat symplectic vector bundles on $\mathbb{S}$, with a monodromy invariant line in a fiber near each puncture, and a non-zero vector in the fiber at each corner. Vectors at the corners of each boundary interval are non-collinear.



Key feature: the gluing map. Given a decorated surface $\mathbb{S}$ and its disjoint boundary intervals I, I', get a new decorated surface $\mathbb{S}'$ by gluing I and I':



Here $\mathbb{S}$ can be disconnected. We define the gluing map of moduli spaces:

$$\gamma_{\mathrm{I},\mathrm{I}'} : \mathcal{P}_{\mathrm{G},\mathbb{S}} \longrightarrow \mathcal{P}_{\mathrm{G},\mathbb{S}'}.$$

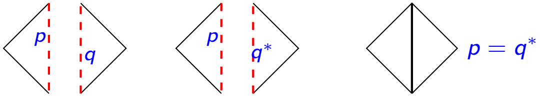
Need pinnings - analog of frames in a vector space.

Example. A pinning for PGL_n is a projective basis in a vector space V_n of dimension n, given by bases up to rescaling: $(e_1, ..., e_n) \sim (te_1, ..., te_n)$.

(A generic pair $(\mathrm{A}_1, \mathrm{A}_2)$ of decorated flags has a Cartan group valued invariant $h(\mathrm{A}_1, \mathrm{A}_2) \in \mathrm{H}$. A pinning p: a pair $(\mathrm{A}_1, \mathrm{A}_2)$ with $h(\mathrm{A}_1, \mathrm{A}_2) = 1$. The dual pinning is $p^* := (\mathrm{A}_2, \mathrm{A}_1)$).

The group G acts freely transitively on pinnings. A generic pair $(\mathrm{A}_1, \mathrm{A}_2)$ determines a unique pinning. So we have pinnings at boundary intervals.

The gluing map is defined by gluing a $\mathrm{G}-$local system on $\mathbb{S}$ into the new one on $\mathbb{S}'$ by matching the pinning on the interval I with the dual pinning on I':



Discrete symmetries of the moduli space $\mathcal{P}_{\mathrm{G},\mathbb{S}}$:

- The mapping class group $\Gamma_{\mathbb{S}}$ of $\mathbb{S}$.
- For each puncture on $\mathbb{S}$, the Weyl group of G (birational).
- For each boundary component of $\mathbb{S}$, the braid group of G.
- The group $\mathrm{Out}(\mathrm{G})$ of outer automorphisms of G.

The semi-direct product $\Gamma_{\mathrm{G},\mathbb{S}}$ of these groups acts on $\mathcal{P}_{\mathrm{G},\mathbb{S}}$.

Theorem A. Let G be a split semi-simple adjoint group $/\mathbb{Q}$. Then the space $\mathcal{P}_{\mathrm{G},\mathbb{S}}$ has a $\Gamma_{\mathrm{G},\mathbb{S}}$-equivariant cluster Poisson structure.

Vladimir Fock - AG, 2003: $\mathrm{G} = \mathrm{PGL}_m$, with $\Gamma_{\mathbb{S}}-$equivariance only.
Ian Le, 2016: same for classical G & G_2, case by case study.
Linhui Shen - AG, 2019 - any G, more flexible even for PGL_m.

A cluster Poisson structure on a variety $\mathcal{X}$ is determined by a single rational coordinate system $\{X_i\}$ and a matrix $\varepsilon_{ij} = -\varepsilon_{ji} \in \mathbb{Z}$.
It provides a Poisson structure at an open part of $\mathcal{X}$:

$$\{X_i, X_j\} := \varepsilon_{ij} X_i X_j.$$

A Poisson cluster $\mathbf{c} = (\{X_i\}, \varepsilon_{ij})$ can be mutated in the direction of any X_k to a new Poisson cluster $\mathbf{c}' = (\{X_i'\}, \varepsilon_{ij}')$. Mutations repeated ad infinitum. (Similar but different than Fomin-Zelevinsky cluster algebras: (i) Mutation rules for the coordinates X_i are different. (ii) Cluster algebras typically do not have any Poisson structure equivariant under the action of the cluster modular group).

The existence of rational coordinates $\{X_i\}$ forces $\mathcal{X}$ to be rational.
The matrix ε_{ij} can be arbitrary. Yet if a group Γ acts on $\mathcal{X}$, the $\Gamma-$equivariance of cluster Poisson structure is a strong condition on ε_{ij}: for any $\gamma \in \Gamma$, the cluster $\gamma^*(\mathbf{c})$ is obtained by mutations from $\mathbf{c}$.

A $\Gamma-$equivariant cluster Poisson structure has two crucial applications:
(i) A $q-$deformation $\mathcal{O}_q(\mathcal{X})$ of the algebra of functions on $\mathcal{X}$, $q \in \mathbb{C}^\times$.
(ii) Cluster quantization, given by canonical $\Gamma-$equivariant representation of

the algebra $\mathcal{O}_q(\mathcal{X})$ by unbounded operators in a Hilbert space.

Non-commutative q−deformation of cluster Poisson varieties.
A Poisson cluster $\mathbf{c} = (\{X_i\}, \varepsilon_{ij}) \longrightarrow$ quantum torus algebra:

$$\mathcal{O}_q(\mathrm{T}_{\mathbf{c}}) := \frac{\mathbb{C}\langle X_i, X_i^{-1}\rangle}{X_iX_j = q^{2\varepsilon_{ij}}X_jX_i}, \quad q \in \mathbb{C}^\times.$$

A mutation $\mathbf{c} \xrightarrow{X_k} \mathbf{c}' \longrightarrow$ isomorphism of fraction fields of $\mathcal{O}_q(\mathrm{T}_{\mathbf{c}'})$ and $\mathcal{O}_q(\mathrm{T}_{\mathbf{c}})$. To define it, recall the quantum dilogarithm power series , also known as q-Pochhammer symbol, q-exponential, and q-Gamma function:

$$\Psi_q(X) := \Big((1+qX)(1+q^3X)\dots\Big)^{-1}.$$

Then $X_i' := \mathcal{M} \circ \mathcal{A}(X_i)$, where $\mathcal{M}$ is a simple monomial transformation,

$$\mathcal{A} : X_i \longmapsto \Psi_q(X_k)X_i\Psi_q(X_k)^{-1} \text{ – a birational automorphism (!).} \qquad (1)$$

In the $q = 1$ limit get the cluster Poisson mutation formulas.

Definition. The algebra $\mathcal{O}_q(\mathcal{X})$ consists of quantum Laurent polynomials $F(X_1, ..., X_n) \in \mathrm{T}_{\mathbf{c}}$ such that for any sequence of mutations $\nu : \mathbf{c}' \to ... \to \mathbf{c}$, the mutated quantum rational function $\nu^*\mathrm{F}$ is a quantum Laurent polynomial in the coordinates $\{X_i'\}$ for the cluster $\mathbf{c}'$.

Faddeev: "*Quantization is an art rather than a functor*".

Key point. For the adjoint G we get a functor from the category of decorated surfaces, with inclusions and gluing maps as morphisms:

$$\mathcal{O}_{\mathrm{G};q} : \text{decorated surfaces } \mathbb{S} \longrightarrow \text{quantum algebras } \mathcal{O}_q(\mathcal{P}_{\mathrm{G},\mathbb{S}}). \qquad (2)$$

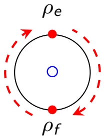
Functoriality for $\mathrm{Diff}(\mathbb{S})$ = the group $\Gamma_{\mathbb{S}}$ acts on $\mathcal{O}_q(\mathcal{P}_{\mathrm{G},\mathbb{S}})$.
Weyl, Braid, $\mathrm{Out}(\mathrm{G})$ groups action = symmetries of the functor.
Functoriality for inclusions, say $\mathbb{S} - \{\text{a loop}\} \subset \mathbb{S}$, are nontrivial.
And $\mathcal{O}_{\mathrm{G};q}$ is clearly a functor for the gluing maps:

$$\gamma : \mathbb{S} \longrightarrow \mathbb{S}' \quad \longrightarrow \quad \gamma^* : \mathcal{O}_q(\mathcal{P}_{\mathrm{G},\mathbb{S}'}) \longrightarrow \mathcal{O}_q(\mathcal{P}_{\mathrm{G},\mathbb{S}}).$$

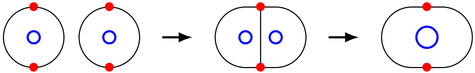
We define the "true space" $\mathrm{Loc}_{\mathrm{G},\mathbb{S}}$ of $\mathrm{G}-$local systems on $\mathbb{S}$ by forgetting the flags at punctures. Its quantum algebra of functions is defined using the cluster nature of the action of the Weyl groups:

$$\mathcal{O}_q(\mathrm{Loc}_{\mathrm{G},\mathbb{S}}) := \mathcal{O}_q(\mathcal{P}_{\mathrm{G},\mathbb{S}})^{W^n}; \qquad n := \#\{\text{punctures on } \mathbb{S}\}.$$

Example 1: The dual Poisson Lie group G^* is identified with the moduli space $\mathrm{Loc}_{\mathrm{G},\odot_{\mu=1}}$ for the punctured disc $\odot$ with two corners e, f:

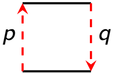


The composition law is given by gluing, followed by encircling:

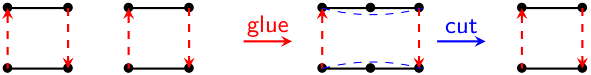


(Invariants $\rho_e, \rho_f \in \mathrm{H}$ at corners; Impose condition $\mu := \rho_e w_0(\rho_f) = 1$).

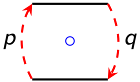
Example 2: The Poisson Lie group G = The moduli space $\mathrm{Loc}_{\mathrm{G},\square}$:



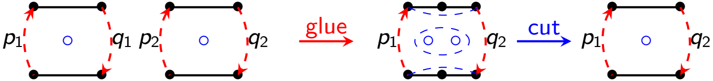
The composition law by gluing and cutting:



Example 3: The Drinfeld double $\mathbb{D}(\mathrm{G})$ of G = The space $\mathrm{Loc}_{\mathrm{G},\boxdot}$:



The composition law:



Note the crucial role of boundary data in all examples. Without it, moduli spaces can be too small/trivial. Boundary data is also crucial for the gluing.

Theorem. We have canonical isomorphism of Poisson Lie groups:

$$\mathrm{G}^* = \mathrm{Loc}_{\mathrm{G},\odot}/\mu = 1.$$

So the algebra $\mathcal{O}_q(\mathrm{Loc}_{\mathrm{G},\odot})/\mu = 1$ becomes a Hopf algebra.

Conjecture (V.Fock-AG, 06) $\exists$ canonical Hopf algebra isomorphism:

$$U_q(\mathfrak{g}) = \mathcal{O}_q(\mathrm{Loc}_{\mathrm{G},\odot})_{\mu=1}.$$

The Hopf algebra $\mathcal{O}_q(\mathrm{Loc}_{\mathrm{G},\odot})_{\mu=1}$ is defined geometrically. But where are the Chevalley generators $\mathbf{E}_i, \mathbf{F}_i, \mathbf{K}_i$? Use boundary data again!

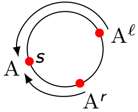
Canonical functions on the space $\mathcal{P}_{\mathrm{G},\mathbb{S}}$ (Linhui Shen - AG 2013).
If the center of G is trivial, the space $\mathcal{A}$ parametrises pairs $(\mathrm{U}_\mathrm{A}, \psi_\mathrm{A})$ where U_A is a max. unipotent subgroup of G, and $\psi_\mathrm{A} : \mathrm{U}_\mathrm{A} \longrightarrow \mathrm{A}^1$ a nondegenerate character. We decompose it as

$$\psi_\mathrm{A} = \sum_{\alpha_i} \psi_{\mathrm{A},\alpha_i}, \qquad \alpha_i \text{ are simple positive roots.}$$

Given $(\mathrm{A}, \mathrm{B}_1, \mathrm{B}_2) \in \mathcal{A} \times \mathcal{B} \times \mathcal{B}$ with generic $(\mathrm{A}, \mathrm{B}_i)$, $\exists! u \in \mathrm{U}_\mathrm{A}$ such that $u \cdot (\mathrm{A}, \mathrm{B}_1) = (\mathrm{A}, \mathrm{B}_2)$. Define potentials

$$\mathcal{W}_{\alpha_i}(\mathrm{A}, \mathrm{B}_1, \mathrm{B}_2) := \psi_{\mathrm{A},\alpha_i}(u).$$

Potentials at the corners. A corner $s \in \mathbb{S}$ carries a decorated flag A_s,



and decorated flags $\mathrm{A}^{\ell}, \mathrm{A}^{r}$ transported from the closest left and right corners, projecting to flags $\mathrm{B}^{\ell}, \mathrm{B}^{r}$ - works even for boundary components with a single corner. We define potentials

$$\mathcal{W}_{s,i} := \mathcal{W}_{\alpha_i}(\mathrm{A}, \mathrm{B}^{\ell}, \mathrm{B}^{r}).$$

Projections to H. Each corner $s \in \mathbb{S}$ provides canonical map

$$\rho_s : \mathcal{P}_{\mathrm{G},\mathbb{S}} \longrightarrow \text{the Cartan group } \mathrm{H}, \qquad \rho_s := h(\mathrm{A}^{r}, \mathrm{A}).$$

So get canonical functions at the corners:

$$\mathcal{K}_{s,i} := \rho_s^*(\text{simple positive root } \alpha_i).$$

Theorem B. (L.Shen-AG 2019) 1) For a corner $s \in \mathbb{S}$, the functions $\mathcal{W}_{s,i}$ and $\mathcal{K}_{s,i}^{\pm 1}$ give rise to canonical quantum algebra elements

$$\mathbf{W}_{s,i}, \ \mathbf{K}_{s,i}^{\pm 1} \ \in \mathcal{O}_q(\mathrm{Loc}_{\mathrm{G},\mathbb{S}}).$$

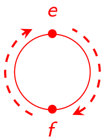
They provide an injective map, (here $\mathfrak{b}$ is a Borel subalgebra in $\mathfrak{g}$):

$$\kappa_s : \mathrm{U}_q(\mathfrak{b}) \hookrightarrow \mathcal{O}_q(\mathrm{Loc}_{\mathrm{G},\mathbb{S}}).$$

$$\mathbf{E}_i \longmapsto \mathbf{W}_{s,i}, \qquad \mathbf{K}_i \longmapsto \mathbf{K}_{s,i}.$$

2) If corners $s, t \in \mathbb{S}$ are not at the same boundary interval, then κ_s, κ_t commute. Otherwise: explicit relations between them.

3) For a boundary component $\pi \subset \mathbb{S}$ with <u>just two</u> corners $\{e, f\}$:



we get an injective map of algebras:

$$\kappa_\pi : \ \mathrm{U}_q(\mathfrak{g}) \hookrightarrow \mathcal{O}_q(\mathrm{Loc}_{\mathrm{G},\mathbb{S}})_{\mu=1};$$
$$\mathbf{E_i} \longmapsto \mathbf{W}_{e,i}, \quad \mathbf{F_i} \longmapsto \mathbf{W}_{f,i^*}, \quad \mathbf{K_i} \longmapsto \mathbf{K}_{e,i}.$$

It identifies Lusztig's braid group action with ours on $\mathrm{Loc}_{\mathrm{G},\mathbb{S}}$.
4) The action of gluing maps on generators is given by simple explicit formulas, reproducing the coproducts. In particular, get an injective map of <u>Hopf algebras</u>

$$\kappa_{\mathrm{G}} : U_q(\mathfrak{g}) \hookrightarrow \mathcal{O}_q(\mathrm{Loc}_{\mathrm{G},\odot})_{\mu=1}.$$

The map κ_{PGL_m} was found by G.Schrader-A.Shapiro (2016).

Theorem (Linhui Shen, 22) For simply-laced G, κ_{G} is an isomorphism.

$\{$Quantum groups$\} \subset \{$quantised algebras of functions on $\mathrm{Loc}_{\mathrm{G},\mathbb{S}}\}$;

Quantum symmetries – induced by the gluing maps.

QUANTIZATION AND REPRESENTATION THEORY

The **most interesting and important application** of **cluster Poisson structure** is **quantization** of cluster Poisson varieties:

We assign to a cluster Poisson variety $\mathcal{X}$:
Langlands modular double algebra: for a Planck constant $\hbar \in \mathbb{C}^\times$:

$$\mathcal{A}_\hbar(\mathcal{X}) := \mathcal{O}_q(\mathcal{X}) \otimes_{\mathbb{C}} \mathcal{O}_{q^\vee}(\mathcal{X}^\vee), \quad q = e^{i\pi\hbar}, \quad q^\vee = e^{i\pi/\hbar}.$$

Cluster modular groupoid Γ of $\mathcal{X}$: objects - Poisson clusters $\mathbf{c}$; morphisms - generated by mutations & isomorphisms of clusters.

A Poisson cluster $\mathbf{c} = (\{X_i\}, \varepsilon_{ij}) \longrightarrow$ Heisenberg Lie algebra $\mathrm{Heis}_{\mathbf{c}}$:

$$\{x_i, x_j\} = 2\pi i\hbar\varepsilon_{ij}.$$

It has canonical representation by unbounded operators $\widehat{x}_k$ in a Hilbert space $\mathcal{H}_{\mathbf{c}}$, selfadjoint if $\hbar > 0$. The algebra $\mathcal{A}_\hbar(\mathcal{X})$ acts by

$$X_k := e^{\widehat{x_k}}, \quad X_k^\vee := e^{\widehat{x_k}/\hbar}$$

on the cluster Schwartz space $\mathcal{S}_{\mathbf{c}} \hookrightarrow \mathcal{H}_{\mathbf{c}}$.
A mutation $\mathbf{c} \xrightarrow{k} \mathbf{c}'$ gives unitary intertwiner

$$\mathcal{I}_{\mathbf{c}\to\mathbf{c}'} := (\text{Simple operator}) \circ \Phi_\hbar(\widehat{x}_k) : \mathcal{H}_{\mathbf{c}} \longrightarrow \mathcal{H}_{\mathbf{c}'}.$$

Here $\Phi_\hbar(z)$ is the quantum dilogarithm function (Barnes, 1899, rediscovered by Baxter, Faddeev, ...):

$$\Phi_\hbar(z) := \exp\Big(-\frac{1}{4}\int_\Omega \frac{e^{-ipz}}{\mathrm{sh}(\pi p)\mathrm{sh}(\pi\hbar p)}\frac{dp}{p}\Big). \\ = \frac{\Psi_q(e^z)}{\Psi_{q^\vee}(e^{z/\hbar})}. \quad (3)$$

The power seris $\Psi_q(X)$ converge only if $|q| < 1$. In our story $|q| = 1$. But the ratio has wonderful analitic properties.
Related to the dilogarithm function $\mathrm{Li}_2(z)$ if $\hbar \to 0$, or $\hbar \in \mathbb{Q}$.

Theorem C. Any cluster Poisson variety $\mathcal{X}$ can be quantized:

- Each cluster $\mathbf{c} \longrightarrow$ a Hilbert space $\mathcal{H}_\mathbf{c}$ & representation of $*-$algebra $\mathcal{A}_\hbar(\mathcal{X})$, $\hbar > 0$, or $|\hbar| = 1$, in the cluster Schwartz space $\mathcal{S}_\mathbf{c}$.
- Mutations $\longrightarrow$ unitary maps, preserve Schwartz spaces, intertwine the action of $\mathcal{A}_\hbar(\mathcal{X})$, provide a unitary representation of groupoid Γ.

CLUSTER REPRESENTATION THEORY.

Cluster quantization provides constructionj of the principal series representations of the $*-$algebra $\mathcal{A}_\hbar(\mathcal{P}_{\mathrm{G},\mathbb{S}})$.
Cluster feature: no preferred vector spaces for representations. Rather groupoid of vector spaces $\{\mathcal{H}_\mathbf{c}\}$ related by quantum intertwiners.
Generalized tensor product: induced by gluing and cutting maps.
For a Poisson Lie group $\mathcal{P}_{\mathrm{G},\mathbb{S}} \to$ "continuous monoidal category".

Classical Representation Theory versus Quantum.

Classical. Principal series unitary representations V_λ of the Lie group $\mathrm{G}_\mathbb{R}$ are parametrised by ($\rho/2-$shifted) characters λ of $\mathrm{H}(\mathbb{R})$, and have a model:

$$L_2(\mathcal{A}) = \int V_\lambda d\lambda. \qquad \mathcal{A} = \mathrm{G}/\mathrm{U}.$$

Representations V_λ and $V_{w\circ\lambda}$ of G are equivalent. The equivalence is given by the Gelfand-Graev unitary intertwiners (1973).
The model for the tensor products:

$$\int (V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}) d\lambda = L_2(\mathcal{A}^n). \tag{4}$$

The model for the tensor product invariants:

$$\int (V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})^{\mathrm{G}} d\lambda = L_2(\mathrm{Conf}_n(\mathcal{A})), \qquad \mathrm{Conf}_n(\mathcal{A}) = \mathrm{G}\backslash\mathcal{A}^n. \tag{5}$$

Note that for a convex $n-$gon $\mathbb{P}_n$ we have $\mathrm{Loc}_{\mathrm{G},\mathbb{P}_n} = \mathrm{Conf}_n(\mathcal{A})$.

Quantum. Cluster Poisson quantization of $\mathrm{Loc}_{\mathrm{G},\odot}/\mu = 1$ gives the principal series $*-$representations $\mathbf{V}_\lambda$ of $U_q(\mathfrak{g})$, parametrised by $\alpha \in \mathfrak{h}^*_\mathbb{R}$.
The Weyl and Braid groups act by quantum dilogarithm unitary intertwiners.
The $W-$action is the quantum analog of Gelfand-Graev intertwiners.

Claim. Modular Functor Conjecture implies that tensor product invariants are realized via $S^2 - \{p_1, ..., p_n\}$:

$$\int \Big(\mathbf{V}_{\lambda_1} \otimes \ldots \otimes \mathbf{V}_{\lambda_n} \Big)_{U_q(\mathfrak{g})} d\lambda \stackrel{?}{=} \mathcal{H}_{\mathrm{G}, S^2 - \{p_1, \ldots, p_n\}} \tag{6}$$

They carry a unitary action of the spherical braid group.

Yet so far quantum and classical representations looks unrelated.

Quasiclassical limit: Consider the topological double $\mathcal{S}_{\mathcal{D}} = \mathbb{S} * \mathbb{S}^{\circ}$ of $\mathbb{S}$.

Figure: The topological double of a hexagon is a sphere with 6 punctures.

Claim. The principal series representations of the $*$-algebra $\mathcal{O}_q(\mathcal{P}_{\mathrm{G}, \mathcal{S}_{\mathcal{D}}})$ can be realized in $L_2(\mathcal{P}_{\mathrm{G}, \mathbb{S}})$.
In particular, $\mathcal{O}_q(\mathcal{P}_{S^2 - \{p_1, \ldots, p_n\}})$ is realized in $L_2(\mathrm{Conf}_n(\mathcal{A}))$.
So tensor product invariants for $U_q(\mathfrak{g})$ realized in the same space as for $\mathrm{G}_{\mathbb{R}}$:

$$\begin{aligned} &\int \Big(\mathbf{V}_{\lambda_1} \otimes \ldots \otimes \mathbf{V}_{\lambda_n} \Big)_{U_q(\mathfrak{g})} d\lambda \stackrel{?}{=} \mathcal{H}_{\mathrm{G}, S^2 - \{p_1, \ldots, p_n\}} \stackrel{\sim}{=} L_2(\mathrm{Conf}_n(\mathcal{A})). \\ &\int \Big(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \Big)_{\mathrm{G}_{\mathbb{R}}} d\lambda = L_2(\mathrm{Conf}_n(\mathcal{A})). \end{aligned} \tag{7}$$

Here is another interpretation of space (**??**):

Conjectural continuous analog of Kazhdan-Lusztig's theory. A Lie algebra $\mathfrak{g}$ gives rise to the W-algebra $\mathcal{W}_{\mathfrak{g}}$, genarlizing the Virasoro Lie algebra. It has series of unitary highest weight modules $\mathbb{V}_{\lambda}$, parametrized by $\alpha \in \mathfrak{h}^*_{\mathbb{R}}$.

(1) Given a complex curve Σ with n punctures, and $\mathcal{W}_{\mathfrak{g}}$-modules $\mathbb{V}_{\lambda_k}$ at the punctures p_k, we have the space of coinvariants:

$$(\mathbb{V}_{\lambda_1} \otimes \ldots \otimes \mathbb{V}_{\lambda_n})_{\Sigma, \mathcal{W}_{\mathfrak{g}}} .$$

(2) For a surface S with n punctures, the cluster quantization of $\mathrm{Loc}_{\mathrm{G},S}$ gives a representation of the $*-$algebra $\mathcal{A}_{\hbar}(\mathrm{Loc}_{\mathrm{G},S})$ in the Schwartz space $\mathcal{S}(\mathrm{Loc}_{\mathrm{G},S})_{\alpha}$, were $\alpha = (\alpha_1, ..., \alpha_n)$, with $\alpha_k \in \mathfrak{h}^*_{\mathbb{R}}$ at puncture p_k.

Conjecture. Let Σ be a punctured Riemann surface, and S its underlying topological surface. Then there exists canonical non-degenerate pairing

$$(\mathbb{V}_{\lambda_1} \otimes \ldots \otimes \mathbb{V}_{\lambda_n})_{\Sigma, \mathcal{W}_{\mathfrak{g}}} \bigotimes \mathcal{S}(\mathrm{Loc}_{\mathrm{G},S})_{\alpha} \longrightarrow \mathbb{C},$$

which is continuous on $\mathcal{S}(\mathrm{Loc}_{\mathrm{G},S})_{\alpha}$. So it induces a linear embedding

$$(\mathbb{V}_{\lambda_1} \otimes \ldots \otimes \mathbb{V}_{\lambda_n})_{\Sigma, \mathcal{W}_{\mathfrak{g}}} \hookrightarrow \mathcal{S}^*(\mathrm{Loc}_{\mathrm{G},S})_{\alpha}. \quad (8)$$

It maps the highest weight vector to the Toda theory conformal block.

So the following three spaces should be "almost the same":
(0) Cluster Poisson quantization space $\mathcal{S}(\mathrm{Loc}_{\mathrm{G},S})$ for $\mathrm{Loc}_{\mathrm{G},S}$.

(1) Coinvariants of principal series rep's of W-algebras $W_{\mathfrak{g}}$ on a curve Σ.
(2) Coinvariants of tensor products of principal series rep's of $U_q(\mathfrak{g})$.

The group $\Gamma_{\mathrm{G},\mathbb{S}}$ and the $*-$algebra $\mathcal{A}_\hbar(\mathrm{Loc}_{\mathrm{G},S})$ act on all of them.