THE CLASSICAL POLYLOGARITHMS, ALGEBRAIC K-THEORY AND $\zeta_F(n)$

A.B. Goncharov

Dedicated to the memory of Larry Corwin

1. Introduction

The classical polylogarithms are defined by the following absolutely convergent series in the unit disc $|z| \leq 1$

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} \tag{1}$$

For example $Li_1(z) = -\log(1-z)$. The differential equation

$$d\, Li_n(z) \;=\; Li_{n-1}(z)\frac{dz}{z} \tag{2}$$

provides an inductive definition of polylogarithms as multivalued analytical functions on $\mathbb{C}P^1 \backslash \{0, 1, \infty\}$:

$$Li_n(z) := \int_0^z Li_{n-1}(w)\frac{dw}{w} \tag{3}$$

The classical polylogarithms were invented in correspondence of Leibniz with J. Bernoulli ([Le]). On November 9, 1696 Leibniz wrote a letter to J. Bernoulli with the formula

$$\sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \left(\int_0^x \frac{dt}{1-t}\right)\frac{dx}{x} \tag{4}$$

On December 1, 1696, Bernoulli informed Leibniz that he had found an analogous formula

$$\sum_{k=1}^{\infty} \frac{1}{k^n} = -\int_0^1 \cdots \int_0^{t_2} \frac{dt_1}{1-t_1} \circ \frac{dt_2}{t_2} \circ \cdots \circ \frac{dt_n}{t_n} \tag{5}$$

They were interested in the summation of series (5) but never succeeded. A few decades later Euler computed numbers (5) for even n and studied the dilogarithm function (4). In the 19th century L. Dirichlet and R. Dedekind discovered a generalization of series (5) for any number field F : zeta function $\zeta_F(s)$. I think that all of these mathematicians would have been pleased to know that according to a conjecture of D. Zagier [Z1], for any number field F, $\zeta_F(n)$ should be expressed by values of the n-logarithm at (complex embedding of) elements of the same field F.

I. M. Gelfand et al. (eds.), *The Gelfand Mathematical Seminars, 1990–1992*
© Birkhäuser Boston 1993

In this article I will explain what this conjecture says and why it is true for $n = 2, 3$. I will also discuss the role of classical polylogarithms in algebraic K-theory and hyperbolic geometry.

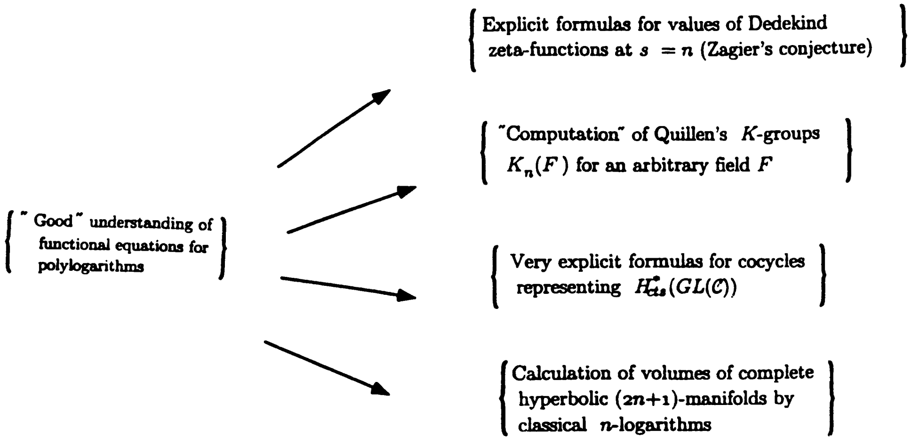
2. Functional Equations for Polylogarithms

The logarithm $\log z$ has a single-valued version $\log |z|$ that satisfies a functional equation

$$\log|xy| = \log|x| + \log|y|.$$

Moreover, a continuous function $f(z)$ satisfying the equation $f(z_1 \cdot z_2) = f(z_1) + f(z_2)$ is proportional to $\log|z|$.

The aim of this paper is to demonstrate that



3. The Dilogarithm

It was investigated widely by Spence (1807), Abel (1827), Kummer (1840), Lobachevsky, Hill, Rogers, Ramanujan, ... The most important discovery of this period was the functional equation (rediscovered many times). We will present it in a form found by Abel.

Theorem 1 (The 5-term relation). *Let* $1 > x > y > 0$. *Then*

$$Li_2(x) - Li_2(y) + Li_2(y/x) - Li_2\left(\frac{1 - x^{-1}}{1 - y^{-1}}\right)$$

$$+ Li_2\left(\frac{1-x}{1-y}\right) = \frac{\pi^2}{6} - \log x \cdot \log\frac{1-x}{1-y}\,. \tag{6}$$

Note that arguments of all function in this formula lie between 0 and 1, so the corresponding values are well-defined. Today it is not so easy to reconstruct reasons for investigation of functional equations for the Dilogarithm in the 19th century. I think that at least for Abel the reason was his famous

Abel's Theorem. *Let*

$$C = \{x,y|f(x,y)=0\}, \quad D_t = \{x,y|g(x,y,t)=0\}$$

are algebraic curves in $\mathbb{C}P^2$. *Set* $\{P_i(t)\} := \{C \cap D_t\}$ *Then*

$$\sum_i \int_{P_0}^{P_i(t)} z(x,y)dx = R(t) + \log S(t) \tag{7}$$

where $z(x,y)$ *is a polynomial,* $\int_{P_0}^{P_i(t)}$ *is an integral along a path on a curve and* $R(t)$, $S(t)$ *are some rational functions.*

Note that each summand $\int_{P_0}^{P_i(t)} z(x,y)dx$ is, of course, a transcendental function on t. (An excellent modern account of Abel's Theorem can be found in [Gr].) The functional equation (6) clearly looks like a generalization of Abel's formula (7): instead of an Abelian integral $\int_{P_0}^{P_i(t)} z(x,y)dx$ we have the simplest example of an iterated integral

$$Li_2(z) = -\int_0^z \frac{dx}{1-x} \circ \frac{dx}{x} := -\int_0^z \left(\int_0^t \frac{dx}{1-x}\right)\frac{dt}{t}$$

while the right-hand side of (6) is a product of logarithms. During the 20th century up to the middle 70's the only enthusiast of polylogarithms was Leonard Lewin (L). Then surprisingly the Dilogarithm appears in works of

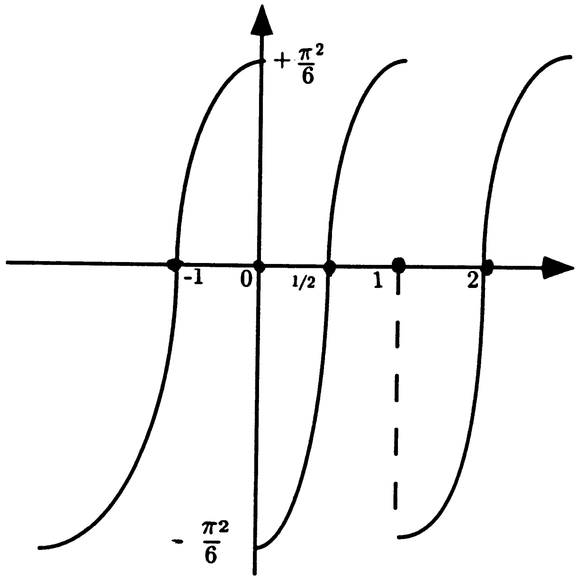
a) A.M. Gabrielov, I.M. Gelfand and M.V. Losik [GGL] on the combinatorial formula for the first Pontryagin class

b) D. Wigner on continuous cohomology of $GL_2(\mathbb{C})$

c) S. Bloch $[Bl1-2]$ on algebraic K-theory and values of zeta-functions at $s=2$.

The function $\phi_2(x)$ considered by Gabrielov, Gelfand and Losik is a version of the Dilogarithm. It can be characterized by the following properties: $\phi_2(x)$ is a function of one *real* variable, smooth an $RP^1\backslash\{0,1,\infty\}$

$$d\phi_2(x) = \frac{\log|x|}{1-x} - \frac{\log|1-x|}{x}, \quad \phi_2(-1) = \phi_2(1/2) = \phi_2(2) = 0\,.$$



It turns out that $\phi_2(x)$ is discontinuous at $x = 0, 1, \infty$:

$$\lim_{x \nearrow 0} \phi_2(x) = \lim_{x \nearrow 1} \phi_2(x) = \lim_{x \to -\infty} \phi_2(x) = +\frac{\pi^2}{6}$$

$$\lim_{x \searrow 0} \phi_2(x) = \lim_{x \searrow 1} \phi_2(x) = \lim_{x \to -\infty} \phi_2(x) = -\frac{\pi^2}{6}.$$

If $0 < x < 1$ then

$$\frac{1}{2}\phi_2(x) = Li_2(x) - \frac{1}{2}\log x \cdot \log(1-x) - \frac{\pi^2}{12}. \tag{8}$$

It turns out that

$$\phi_2(x) = -\phi_2(1-x) = -\phi_2\left(\frac{1}{x}\right).$$

Now let $x_0, \ldots, x_3$ be 4 distinct points on RP^1 and let

$$r(x_0, \ldots, x_3) = \frac{(x_0 - x_2)(x_1 - x_3)}{(x_0 - x_3)(x_1 - x_2)}$$

be the cross-ratio. Then for 5 distinct points $x_0, \ldots, x_4$ on RP^1, one has

$$\sum_{i=0}^{4} (-1)^i \phi_2(r(x_0, \ldots, \hat{x}_i, \ldots, x_4)) = \varepsilon \cdot \frac{\pi^2}{6} \tag{9}$$

where $\varepsilon = \pm 1$. The precise value of ε is computed as follows: choose an orientation in R^2 and a 5-tuple of vectors $(l_0, \ldots, l_4)$ that are projected to $(x_0, \ldots, x_4)$. Then $\varepsilon = \pm 1$ if the number of bases (l_α, l_β) in $R^2 (\alpha < \beta)$ with positive orientation is even and -1 in the opposite case. (This definition does not depend on the choice of vectors $(l_0, \ldots, l_4)$). If $1 > x > y > 0$ then the functional equation coincides essentially with the one (7).

Another version of the Dilogarithm was considered by D. Wigner and S. Bloch. They invented the function

$$D_2(z) := \mathrm{Im}Li_2(x) + \arg(1-z) \cdot \log|z| \tag{10}$$

(the Bloch–Wigner function), that is continuous (and in particular single-valued) on $\mathbb{C}P^1$. The 5-term functional equation for $D_2(z)$ is

$$\sum_{i=0}^{4}(-1)^i D_2(r(z_0, \ldots, \hat{z}_i, \ldots, z_4)) = 0; \;\; z_i \neq z_j \in \mathbb{C}P^1 . \tag{11}$$

Let $x \in \mathbb{C}P^1$. D. Wigner discovered that (11) just means that

$$f_3^{(x)}(g_0, \ldots, g_3) := D_2(r(g_0 x, \ldots, g_3 x)), \;\; g_i \in GL_2(\mathbb{C}) \tag{12}$$

is a (measurable) 3-cocycle for the group $GL_2(\mathbb{C})$. Another point $x' \in \mathbb{C}P^1$ gives a cocycle that is canonically cohomologous to the previous one.

Let G be a Lie group, $G^n := \underbrace{G \times \ldots \times G}_{n\,\text{times}}$, $M(G^n)$: the space of measurable functions on G^n. There is a differential

$$d : M(G^n) \to M(G^{n+1})$$
$$(df)(g_1, \ldots, g_{n+1}) = \sum_{i=1}^{n+1}(-1)^i f(g_1, \ldots, \hat{g}_i, \ldots, g_{n+1}) .$$

Then

$$H^*_{(m)}(G, R) := H^{*+1}(\underset{\ldots}{\to} M(G^{n-1})^g \xrightarrow{d} M(G^n)^G \xrightarrow{d} M(G^n)^G \xrightarrow{d} M(G^{n+1})^G \ldots)$$

is the measurable cohomology of the Lie group G. It is known that

$$\dim H^3_{(m)}(GL_2(\mathbb{C}), R) = 1.$$

The cocycle (12) represents a nontrivial cohomology class.

Theorem 2 (S. Bloch, Bl. 2]). *Let $f(z)$ be a measurable function on $\mathbb{C}P^1$ such that $\sum_{i=0}^4(-1)^i f(r(z_0, \ldots, \hat{z}_i, \ldots, z_4)) = 0$. Then $f(z) = \lambda \cdot D_2(z)$.*

Moreover, it turns out that *any* functional equation for $D_2(z)$ is a formal consequence of the 5-term equation (11). (see Section 11 below)

Now let us give a geometrical interpretation of the Bloch–Wigner function. Let H^3 be the Lobachevsky space. Then $\partial H^3 \cong \mathbb{C}P^1$. Denote by $I(z_0, \ldots, z_3)$

the ideal tetrahedron with vertices at points $z_0, \ldots, z_3$ of the absolute ∂H^3. It is clear that

$$\sum_{i=0}^{4}(-1)^i I(z_0, \ldots, \hat{z}_i, \ldots, z_4) = \phi. \tag{13}$$

It is easy to check that $I(z_0, \ldots, z_3)$ has a finite volume vol $(I(z_0, \ldots, z_3))$. So according to Theorem 2 and (13)

$$\text{vol } (I(z_0, \ldots, z_3)) = \lambda \cdot D_2(r(z_0, \ldots, z_3)), \quad \lambda \in \mathbb{R}^*.$$

Any complete hyperbolic 3-manifold can be cut on a finite number of ideal tetrahedrons $I_{z_i} := I(\infty, 0, 1, z_i)$. Therefore its volume is equal to $\Sigma D_2(z_i)$. Note that $D_2(z_i) = -D_2(\bar{z}_i)$. So we can write this sum as $\frac{1}{2}\Sigma(D_2(z_i) - D_2(\bar{z}_i))$. It follows imemdiately from results of Dupont–Sah [DS] and Neumann–Zagier [NZ] that numbers z_i have to satisfy the relation

$$\sum_i ((1 - z_i)) \wedge z_i - (1 - \bar{z}_i) \wedge \bar{z}_i) = 0 \text{ in } (\Lambda^2\mathbb{C}^*)^-.$$

Here $\Lambda^2\mathbb{C}^*$ is the wedge square of the abelian group $\mathbb{C}^*$ and $(\Lambda^*\mathbb{C}^*)^-$ is the subgroup of anti-invariants of the action of complex conjugation.

The relation just means that the sum of the Dehn invariants of the tetrahedrons I_{z_i} is equal to 0. Recall that the Dehn invariant of a finite geodesic tetrahedron is defined as

$$\sum_A l(A) \otimes \alpha_A \in R \otimes R/_{2\pi\mathbf{Z}}$$

where A runs through all edges of length $l(A)$ with dihedral angle α_A. To define the Dehn invariant in the case when the tetrahedron has vertices at absolute, following Thurston, let us delete a horoball around each infinite vertex and for each A an edge ending this vertex the length $l(A)$ is measured only up to the horosphere. The indeterminacy in this definition vanishes because the sum of the angles at the edge ending a vertex at infinity is π.

Example. The Dehn invariant of the ideal tetrahedron I_z is equal to

$$\log|1 - z| \otimes \arg z - \log|z| \otimes \arg(1 - z).$$

4. The Trilogarithm and $\zeta_F(3)$

Set

$$\mathcal{L}_3(z) := \text{Re}(Li_3(z) - Li_2(z) \cdot \log|z| + \frac{1}{3}Li_1(z) \cdot \log^2|z|). \tag{14}$$

Then $\mathcal{L}_3(z)$ is continous on $\mathbb{C}P^1$.

Let $\mathbb{Z}[\mathbb{P}^1_F]$ be a free abelian group generated by symbols $\{z\}$, where z runs through all F-points of $\mathbb{P}^1$. In the case $F = \mathbb{C}$, any real-valued function on $\mathbb{C}P^1$, and in particular $\mathcal{L}_3(z)$, defines a homomorphism

$$\tilde{\mathcal{L}}_3 : \mathbb{Z}[\mathbb{C}P^1] \to \mathbb{R}$$
$$\{z\} \mapsto \mathcal{L}_3(z) \tag{14a}$$

Now let $R_2(F) \subset \mathbb{Z}[P^1_F]$ be a subgroup generated by $\{0\}, \{\infty\}$ and

$$\sum_{i=0}^{4}(-1)^i\{r(x_0, \dots, \hat{x}_i, \dots, x_4)\}, \quad x_i \in P^1_F, \quad x_i \neq x_j\,.$$

Set

$$B_2(F) := \frac{\mathbb{Z}[P^1_F]}{R_2(F)} \tag{15}$$

Let us define a homomorphism

$$\delta_3 : \mathbb{Z}[P^1_F] \to B_2(F) \otimes F^*$$
$$\delta_3 : \{z\} \mapsto \{z\}_2 \otimes z$$
$$\{0\}, \{\infty\} \mapsto 0$$

Here $\{z\}_2$ is the image of $\{z\}$ in $B_2(F)$.

Theorem 3 (Zagier's conjecture [Z1]). *a) Let F be a number field, $[F : \mathbb{Q}] = r_1 + 2r_2, \sigma_j : F \hookrightarrow \mathbb{C}$ are all possible imbeddings of F in $\mathbb{C}$ numbered so that $\overline{\sigma_{r_1+i}} = \sigma_{r_1+r_2+i}$, d_F is the discriminant of F. Then there exist elements $y_1, \dots, y_{r_1+r_2} \in Ker\, \delta_3 \subset \mathbb{Z}[P^1_F]$ such that*

$$q \cdot \zeta_F(3) = \cdot\pi^{3\pi \cdot r_2} \cdot |d_F|^{-\frac{1}{2}} \cdot det|\tilde{\mathcal{L}}_3(\sigma_j(y_i))| \;\; (1 \leq i, j \leq r_1 + r_2) \tag{16}$$

where $q \in \mathbb{Q}^*$

b) for any *elements $y_1, \dots, y_{r_1+r_2} \in Ker\, \delta_3$ formula (16) holds with $q \in \mathbb{Q}$.*

Example 4 $\delta_3\{1\} = \{1\}_2 \otimes 1 = 0$. $\zeta_{\mathbb{Q}}(3) = \mathcal{L}(1)$. But $\mathbb{Q}$ and, more generally, the cyclotomic fields are the only ones for which (16) is easy to check.

5. Zagier's Conjecture

D. Zagier conjectured [Z1] that for any number field F there exist elements $y_1, \dots, y_{d_n} \in \mathbb{Z}[P^1_F]$ such that $(n > 1)$

$$q \cdot \zeta_F(n) = \pi^{n \cdot (r_1 + 2r_2 - d_n)} \cdot |d_F|^{-\frac{1}{2}} \cdot \det|\tilde{\mathcal{L}}_n(\sigma_j(y_i))| \tag{17}$$

where $q \in \mathbb{Q}^*$,

$$d_n = \begin{cases} r_1 + r_2 & \text{for } n : \text{odd}, \\ r_2 & \text{for } n : \text{even}, \quad 1 \leq i, j \leq d_n \end{cases}$$

and for $n > 1$

$$\mathcal{L}(z) := \begin{cases} \mathrm{Re}\, n: & \text{odd} \\ \mathrm{Im}\, n: & \text{even} \end{cases} \left(\sum_{k=1}^{n} \frac{B_k \cdot 2^k}{k!} Li_{n-k}(z) \cdot \log^k |z| \right) \tag{18}$$

is a single-valued version of $Li_n(z)$.

Elements $y_1, \dots, y_{d_n}$ should satisfy an algebraic condition analogous to $\delta_3(y_i) = 0$ in $B_2(F) \otimes F^*$ for $n = 3$.

Example 5. $\zeta_{\mathbb{Q}}(n) = \mathcal{L}_n(1)$, just by definition.

For $n = 2$, formula (17) was proved by Zagier [Z2] and also follows immediately from results of S. Bloch, A. Borel [Bo 1-2] and A. Suslin [S2]. The only general result about $\zeta_F(n)$, $n > 3$ in this direction is the Klingen–Siegel theorem: for totally real fields F (i.e., $r_2 = 0$)

$$\zeta_F(2n) = q \cdot \frac{\pi^{2r_1 \cdot n}}{\sqrt{d_F}}, \quad q \in Q^*$$

that generalizes the Euler formula for $\zeta_{\mathbb{Q}}(2n)$.

The analog of formula (17) for $n = 1$ is the classical Dedekind formula

$$\mathrm{Res}\, \zeta_F(s) = \frac{\pi^{r_2} \cdot 2^{r_1 + r_2} \cdot h}{w \cdot \sqrt{d_F}} \cdot R_1 \tag{19}$$

where h is the class number of the field F, w is the number of roots of 1 in F and R_1 is the *regulator* that is defined as follows. Take a basis of fundamental units $\varepsilon_1, \dots, \varepsilon_{r_1+r_2-1}$ in the free part of the abelian group $\mathcal{O}_K^*$. Then

$$R_1 = |\det(\log|\sigma_i(y_j)|^{a_i})$$

where $1 \le i, j \le r_1 + r_2 - 1$ and $a_i = 1$ for real σ and 2 for complex one.

In a remarkable paper [BD] A.A. Beilinson and P. Deligne proved an analog of statement b) of Theorem 3 for any n. However, the main problem: whether there exist elements $y_i \in \mathbb{Z}[P_F^1]$ such that the corresponding constant q in the left hand side of (17) is *non-zero* (and so there is a formula for $\zeta_F(n)$) remains unsolved.

Now let me present the main ingredients of the proof of Theorem 3.

6. The Borel Regulator

A. Borel defined a homomorphism $\tilde{r}_n : K_{2n-1}(\mathbb{C}) \to R$. Let us recall this definition. One has

$$K_m(F) := \pi_m(BGL(F)^+) \overset{Hurewicz}{\longrightarrow} H_m(BGL(F)^+) = H_m(GL(F)) \quad (20)$$

Now let $F = \mathbb{C}$. There is the canonical pairing

$$H^{2n-1}(GL(\mathbb{C}), R) \times H_{2n-1}(GL(\mathbb{C}), R) \overset{<,>}{\longrightarrow} R$$

There is a subspace

$$H^{2n-1}_{(m)}(GL(\mathbb{C}, R) \subset H^{2n-1}(GL(\mathbb{C}), R)$$

It is known that

$$H^*_{(m)}(GL(\mathbb{C}), R) = \wedge^*_R(c_1, c_3, c_5, \ldots)$$

where $c_{2n-1} \in H^{2n-1}_{(m)}(GL(\mathbb{C}), R)$ are the Borel classes. (The restriction of c_{2n-1} to $GL_m(\mathbb{C})$ is nontrivial for $m \geq n$). So c_{2n-1} defines a homomorphism $H_{2n-1}(GL(\mathbb{C}), R) \to R$ and hence by (20) regulator $\tilde{r}_n$. Let $R(n) := (2\pi i)^n \cdot R \subset \mathbb{C}$. Then one has

$$K_{2n-1}(F) \to \bigoplus_{\mathrm{Hom}(F,\mathbb{C})} K_{2n-1}(\mathbb{C}) \overset{\tilde{r}_n \otimes R(n-1)}{\longrightarrow} [\mathbb{Z}^{\mathrm{Hom}(F,C)} \otimes R(n-1)]$$

where the first arrow is provided by the functoriality of K-groups. It turns out that the image of $K_{2n-1}(F)$ in $\mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C})} \otimes R(n-1)$ is invariant under the complex conjugation, so we get a homomorphism

$$r_n : K_{2n-1}(F) \to [\mathbb{Z}^{\mathrm{Hom}(F,\mathbb{C}} \otimes R(n-1)]^+ \quad (21)$$

This is the Borel regulator.

Theorem 4 ([Bo 1-2]). *Suppose that* $n > 1$. *Then:*

a. *Ker* r_n *is torsion*

b. *Im* r_n *is a lattice*

c. *Covolume* $(Im\, r_n) = q \cdot lim_{s \to 1-n}(s-1+n)^{-d_n} \zeta_F(s)$ *where* $q \in \mathbb{Q}^*$.

The functional equation for $\zeta_F(s)$ shows the right-hand side of (22) is equal up to a nonzero rational factor to

$$\sqrt{|d_F|} \cdot \pi^{-n(r_1+2r_2-d_n)} \cdot \zeta_F(n)$$

Example 5. If $n = 1$ then

$$H_1(GL(F),\mathbb{Z}) := GL(F)/[GL(F),GL(F)] \overset{\det}{\xrightarrow{\sim}} F^* \cong K_1(F),$$

$c_1 \in H^1_{(m)}(GL(\mathbb{C}))$ is represented by a cocycle

$$f_1(g_0, g_1) := \log|\det(g_0^{-1}g_1)| \tag{22}$$

and so $r_1 : \mathbb{C}^* \to R$ is given by formula $z \mapsto \log|z|$.

The analog of Theorem 4 in the case $n = 1$ is the Dedekind theorem (19).

Theorem 4 explains the importance of explicit formulas for cocycles representing the Borel class in $H^*_{(m)}(GL(\mathbb{C}))$

a. A cocycle for the class c_1 is given by the formula (22).

b. A cocycle for the class $c_3 \in H^3_{(m)}(GL_2(\mathbb{C}))$ is given by D. Wigner's formula (12).

7. An Explicit Formula for a Measurable Cocycle Representing the Borel Class $c_5 \in H^5_{(m)}(GL_3(\mathbb{C}))$

Choose a non-zero element $w_3 \in \wedge^3(\mathbb{C}^3)^*$. Let $(l_1, \dots, l_b)$ be a 6-tuple of vectors in generic position in $\mathbb{C}^3$. Set

$$\Delta(l_i, l_j, l_k) :=< w_3, l_i \wedge l_j \wedge l_k >$$
$$r_3'(l_1, \dots, l_6) := \frac{\Delta(l_1,l_2,l_4)\cdot\Delta(l_2,l_3,l_5)\cdot\Delta(l_3.l_1,l_6)}{\Delta(l_1,l_2,l_5)\cdot\Delta(l_2,l_3,l_6)\cdot\Delta(l_3,l_1,l_4)}. \tag{23}$$

It is clear that $r_3'(l_1, \dots, l_6)$ does not depend on the length of vectors l_i and GL_3-invariant. It depends only on the corresponding configurations $(\bar{l}_1, \dots, \bar{l}_6)$ of 6 point in $\mathbb{C}P^2$. Let us define *the generalized cross-ratio*

$$r_3(\bar{l}_1, \dots, \bar{l}_6) := \sum_{\sigma \in S_6} (-1)^{|\sigma|}\{r_3'(\bar{l}_{\sigma(1)}, \dots, \bar{l}_{\sigma(6)})\} \in \mathbb{Z}[P^1_{\mathbb{C}}]. \tag{24}$$

Then

$$\tilde{\mathcal{L}}_3(r_3(\bar{l}_1, \dots, \bar{l}_6)) \tag{25}$$

is a function on configurations of 6 points in $\mathbb{C}P^2$.

Theorem 5 ([G4]). *For any 7 points in generic position $(\bar{l}_1, \dots, \bar{l}_7)$ in $\mathbb{C}P^2$*

$$\sum_{i=1}^{7}(-1)^i\mathcal{L}(r_3(\bar{l}_1, \dots, \hat{l}_i, \dots, \bar{l}_7)) = 0. \tag{26}$$

An interpretation: choose a point $x \in \mathbb{C}P^2$. Then

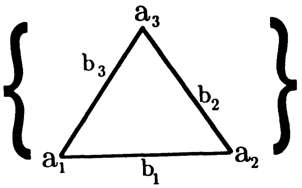
$$f_5^{(x)}(g_0, \ldots, g_5) := \mathcal{L}_3(r_3(g_0 x, \ldots, g_5 x)) \tag{27}$$

is a 5-cocycle of $GL_3(\mathbb{C})$.

Theorem 6 ([G4]. *The cohomology of the cocycle coincides with the Borel class.*

Proof. See proof of Theorem 5.12 in [G4].

Now let me give a geometrical interpretation of the generalized cross-ratio ζ_3. First of all let me note that there is an isomorphism



$$\xrightarrow[s]{\sim} P^1_F \backslash \{0, \infty\}$$

$/PGL_3(F)$

provided by the formula

$$s : (a_1, a_2, a_3, b_1, b_2, b_3) \mapsto \frac{f_1(b_2) \cdot f_2(b_3) f_3(b_1)}{f_1(b_3) \cdot f_2(b_1) \cdot f_3(b_2)} \in F^* . \tag{28}$$

Here $(a_1, a_2, a_3, b_1, b_2, b_3)$ is a 6-tuple of distinct points in P_F^2 such that a_1, a_2, a_3 do not lie on a line and $b_i \in \overline{a_i a_{i+1}}$ (indices modulo 3). In (28) $f_i \in V_3^*$ are some linear functionals such that $f_i(a_i) = f_i(a_{i+1})$. Formula (28) is well-defined because it does not depend on the choice of these functionals and vectors in V_3 represented the points b_i.

For example, $1 \in F^*$ is represented by a configuration where b_1, b_2, b_3 lie on the same line (see Fig. 1) and $-1 \in F^*$ is represented by a configuration where the lines $\overline{a_1 b_2}, \overline{a_2 b_3}$ and $\overline{a_3 b_1}$ intersects in a point. (See Fig. 2)

Now let $(l_1, \ldots, l_6)$ be a generic configuration of 6 points in P^2. Put $a_i := l_i l_{i+3} \cap l_{i-1} l_{i+2}$ $(1 \leq i \leq 3$, indices modulo 6; see Fig. 3). Then $(a_1, a_2, a_3, l_1, l_2, l_3)$ is a configuration of the above type.

Lemma 7. $s(a_1, a_2, a_3, l_1, l_2, l_3) = r_3'(l_1, l_2, l_3, l_4, l_5, l_6)$.

Proof. See proof of Lemma 3.8 in [G4].

It turns out that the function $\tilde{\mathcal{L}}_3(r_3(l_1, \ldots, l_6))$ satisfies another functional equation. Let $(l_1, \ldots, l_7)$ be a generic configuration of 7 points in P^3. Let us denote by $(l_i | l_1, \ldots, \hat{l}_i, \ldots, l_7)$ the configuration of 6 points in P^2 obtained by projection of points l_j, $j \neq i$ with the center at the point l_i. More precisely,

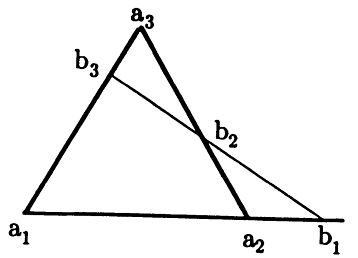


Fig. 1

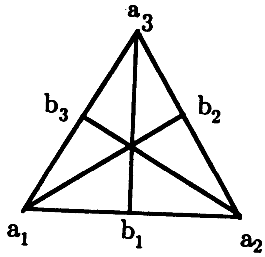


Fig. 2

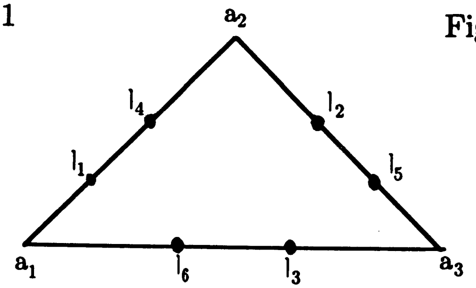


Fig. 3

the set of all lines in P^3 through the point l_i can be identified with P^2 and each point l_j, $j \neq i$ defines a point in this P^2.

Theorem 8 [G4] (The dual 7-term relation). *Let* $(l_1, \dots, l_7)$ *be a generic configuration of 7-points in* $\mathbb{C}P^3$. *Then*

$$\sum_{i=1}^{7} (-1)^i \tilde{\mathcal{L}}_3(r_3(l_i | l_1, \dots, \hat{l}_i, \dots, l_7)) = 0. \tag{29}$$

Proof. See proof of Theorem 3.12 in [G3].

The functional equation (29) can be deduced from the one (28) (see [G4]). However it plays an important role in the proof of Theorem 9 below.

8. A Formula for a Cocycle Representing the Borel Class $c_5 \in H^5_{(m)}(GL_n(\mathbb{C}))$ for any $n \geq 3$

Recall that a p-flag in P^k is a sequence

$$L_\bullet := (L_o, L_1, \dots, L_{p-1})$$

where L_i is an i-dimensional plane in P^k and $L_i \subset L_{i+1}$.

Let us denote by $H_1 * H_2$ the *joining* of planes $H_1, H_2 \subset P^k$. Note that for generic planes H_1, H_2 we have $\dim(H_1 * H_2) = \dim H_1 + \dim H_2 - 1$. By definition $\phi * H = H * \phi = H$. Let us define the generalized cross-ratio of 6 generic $(n-3)$-flags in P_F^{n-1}:

$$r_3^{(n)}(L_\bullet^{(1)}, \ldots, L_\bullet^{(6)}) \in \mathbb{Z}[P_F^1] \tag{30}$$

as follows:

$$r_3^{(n)}(L_\bullet^{(1)}, \ldots, L_\bullet^{(6)}) := \sum_{\substack{j_1+\ldots+j_6=n-2 \\ j_k \geq 0}} (L_{j_1-1}^{(1)} * \ldots * L_{j_1-1}^{(6)} | L_{j_1}^{(1)}, \ldots, L_{j_6}^{(6)}) . \tag{31}$$

Here $(L_{j_1-1}^{(1)} * \ldots * L_{j_1-1}^{(6)} | L_{j_1}^{(1)}, \ldots, L_{j_6}^{(6)})$ is a configuration of 6 points in P^2 obtained by the projection of $L_{jk}^{(k)}$ with the center at $L_{j_1-1}^{(1)} * \ldots * L_{j_6-1}^{(6)}$. More precisely, the set of all planes of dimension $j_1 + \ldots + j_6$ containing $L_{j_1-1}^{(1)} * \ldots * L_{j_6-1}^{(6)}$ forms a projective plane P^2 because of the condition $j_1 + \ldots + j_6 = n-2$ (and the assumption of generic position). Each $L_{jk}^{(k)}$ defines a point on this plane.

For example, the cross-ratio of 6 2-flags in P^3 is given by the formula (see also Fig. 4)

$$r_3^{(4)}(L_\bullet^{(1)}, \ldots, L_\bullet^{(6)}) := \sum_{k=1}^{6} r_3(L_0^{(k)} | L_0^{(1)}, \ldots, L_1^{(k)}, \ldots, L_0^{(6)}) .$$

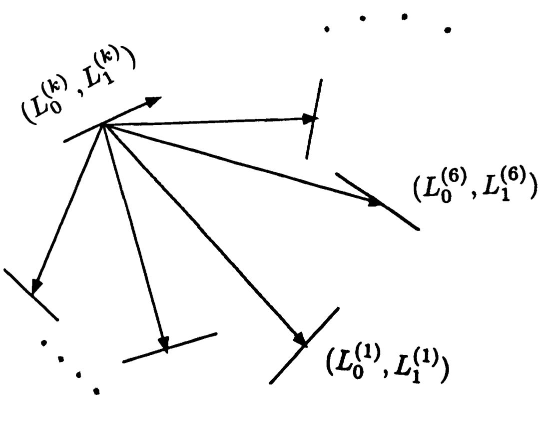


Fig.4

Theorem 9 [G3]. *Choose an $(n-3)$-flag L_0 in $\mathbb{C}P^{m-1}$. Then*

$$\tilde{\mathcal{L}}_3(r_3^{(n)}(g_0 \cdot L_\bullet, \ldots, g_5 \cdot L_\bullet))$$

is a (measurable) 5-cocycle of $GL_n(\mathbb{C})$ *representing the Borel class* $c_5 \in H^5_{(m)}(GL_n(\mathbb{C}))$.

Let me present the proof of the simplest case $n = 4$. We have to prove that

$$\tilde{\mathcal{L}}_3\left(\sum_{j\neq i}\sum_{i=1}^{7}(-1)^i r_3(L_0^{(j)}|L_0^{(1)},\ldots,\widehat{L_0^{(i)}},\ldots,L_1^{(j)},\ldots,L_0^{(7)})\right) = 0. \qquad (32)$$

Applying the 7-term relation for the following configuration of 7 points in P^2 $(L_0^{(j)}|L_0^{(1)},\ldots,L_1^{(j)},\ldots,L_0^{(7)})$ (j is fixed) one can rewrite (32) as

$$\tilde{\mathcal{L}}_3\left(\sum_{j=1}^{7}(-1)^j r_3(L_0^{(j)}|L_0^{(1)},\ldots,\widehat{L_0^{(j)}},\ldots,L_0^{(7)})\right) = 0.$$

But this is just the dual 7-term relation (29).

9. The Trilogarithm is Determined by the 7-term Functional Equation

Let us define a subgroup

$$R_3(F) := \left\{\sum_{i=1}^{7}(-1)^i r_3(l_1,\ldots,\hat{l}_i,\ldots,l_7)\right\}$$

where $(l_1,\ldots,l_7)$ runs through all generic configurations of 7 points in $\mathbb{C}P^2$.

Theorem 10. *Let* $f(z) \in C^\infty(\mathbb{C})$ *be a function satisfying the functional equation* $\tilde{f}(R_3(\mathbb{C})) = 0$, *i.e.,*

$$\sum_{i=1}^{7}(-1)^i \tilde{f}(r_3(l_1,\ldots,\hat{l}_i,\ldots,l_7)) = 0$$

for generic 7-tuple points in CP^2. *Then*

$$f(z) = \lambda\cdot\mathcal{L}_3(z) + \beta\cdot D_2(z)\cdot\log|z|\,.$$

10. Algebraic K-Theory of fields and Classical Polylogarithms: results

Now let F be an arbitrary field. Let us define subgroups $R_i(F) \subset \mathbb{Z}[P^1_F]$ $(i = 1,2,3)$ as the ones generated by the following elements:

$$R_1(F) := (\{x\} + \{y\} - \{xy\};\ x,y \in F^*)$$

$$R_2(F) := (\sum_{i=1}^{5}(-1)^i\{r(x_1,\ldots,\hat{x}_i,\ldots,x_5)\};\ x_i \neq x_j \in P^1_F)$$

$$R_3(F) := (\sum_{i=1}^{7}(-1)^i r_3(l_1,\ldots,\hat{l}_i,\ldots,l_7);\ l_i \in P^2_F)$$

Set

$$B_i(F) := \frac{\mathbb{Z}[P^1_F]}{R_i(F), \{0\}, \{\infty\}}.$$

Then $B_i(F)^\vee := \text{Hom}\,(B_i(F), \mathbb{Z})$ is the group of "abstract i-logarithms," i.e., set-theoretic functions on P^1_F satisfying the functional equation for i-logarithm. We have

$$\begin{aligned} B_1(F) &\overset{\sim}{\to} F^* \\ \{x\} &\mapsto x. \end{aligned}$$

Let us consider the following complexes $B_F(n)$:

$$\begin{aligned} &B_F(3) : B_3(F) \overset{\delta_3}{\to} B_2(F) \otimes F^* \overset{\delta_3}{\to} \Lambda^3 F^* \\ &B_F(2) : B_2(F) \overset{\delta_2}{\to} \Lambda^2 F^* \\ &BF(1) : F^* \end{aligned}$$

where $\delta_2 : \{x\} \mapsto (1-x)\wedge x; \delta_3 : \{x\}_3 \mapsto \{x\}_2 \otimes x; \delta_3 : \{x\}_2 \otimes y \mapsto (1-x)\wedge x \wedge y.$ ($\{x\}_n$ is the projection of $\{x\}$ to $B_n(F), B_i(F)$ placed in degree 1 and δ has degree $+1$. It is clear that $\delta_3^2 = 0$. The homology of these complexes are related to algebraic K-theory as follows:

$$\begin{aligned} H^1(B_F(1)) &\equiv F^* = K_1(F) \\ H^2(B_F(2)) &= K_2(F) \text{ by Matsumoto theorem [Ma]} \\ H^1(B_F(2) \otimes \mathbb{Q}) &= K_3^{\text{ind}_3}(F) \otimes \mathbb{Q} \text{ by [S2-3], see also [Sa]} \\ H^3(B_F(3)) &= K_3^M(F) \text{ by definition of Milnor's } K-\text{theory [M]} \end{aligned}$$

Here

$$K_n^M(F) := \frac{\Lambda^n F^*}{((1-x)\wedge x \wedge \Lambda^{n-2} F^*)}$$

are the Milnor K-groups ([M]). The multiplication in $K_*(F)$ induces a map $m : K_1(F) \times \ldots \times K_1(F) \to K_n(F)$ that factorizes through a map $s : K_n^M(F) \to K_n(F)$:

$$\begin{array}{ccc} F^* \times \ldots \times F^* & \longrightarrow & K_n(F) \\ \searrow & & \nearrow s \\ & K_n^M(F) & \end{array}$$

According to [G2], [G4] there are canonical maps

$$\begin{aligned} K_4(F) &\to H^2(B_F(3)) \\ K_5(F) &\to H^1(B_F(3)). \end{aligned}$$

A.A. Suslin proved ([S1]) that s is injective modulo $(n-1)!$ By definition

$$K_3^{\text{ind}}(F) := \frac{K_3(F)}{s(K_3^M(F))}$$

To formulate a more precise result let me introduce the rank filtration on $K_n(F)$. Recall that

$$K_n(F) := \pi_n(BGL(F)^+)$$

where $BGL(F)^+$ is an H-space such that

$$H_n(BGL(F)^+) = H_n(GL(F)).$$

So by the Milnor–Moore theorem

$$K_n(F) \otimes \mathbb{Q} = \text{Prim } H_n(GL(F), \mathbb{Q})$$

A.A. Suslin proved that the natural map

$$H_n(GL_n(F)) \to H_n(GL(F))$$

is an isomorphism. Therefore there is a filtration on $K_n(F)_{\mathbb{Q}} := K_n(F) \otimes \mathbb{Q}$

$$K_n(F)_{\mathbb{Q}} = K_n^{(0)}(F) \supset \dots$$

$$K_n^{(i)}(F) := H_n(GL_{n-i}(F), \mathbb{Q}) \cap \text{Prim } H_n(GL(F), \mathbb{Q}).$$

Set

$$K_n^{[i]}(F) := \frac{K_n^{(i)}(F)}{K_n^{(i+1)}(F)}.$$

Theorem 11 ([G2], [G4]). *There are canonical maps*

$$K_4^{[1]}(F) \to H^2(B_F(3) \otimes \mathbb{Q})$$
$$K_5^{[2]}(F) \to H^1(B_F(3) \otimes \mathbb{Q}).$$

Conjecture 12. *These maps are isomorphisms.*

Note that A.A. Suslin proved that (see [S1])

$$K_n^{[0]}(F)_{\mathbb{Q}} \cong K_n^M(F)_{\mathbb{Q}}$$

11. Algebraic K-theory of Fields and Classical Polylogarithms: conjectures

Let us define by induction subgroups $\mathcal{R}_n(\mathbb{F}) \subset \mathbb{Z}[P^1_F]$, $n \geq 1$. Set

$$\mathcal{B}_n(F) := \mathbb{Z}[P^1_F]/\mathcal{R}_n(F)$$

$$\mathcal{R}_1(\mathcal{F}) := (\{x\} + \{y\} - \{xy\},\ \ (x, y \in F^*); \{0\}; \{\infty\}).$$

Consider homomorphisms

$$\mathbb{Z}[P^1_F] \xrightarrow{\delta_n} \begin{cases} \mathcal{B}_{n-1}(F) \otimes F^* & : n \geq 3 \\ \wedge^2 F^* & : n = 2 \end{cases}$$

$$\delta_n : \{x\} \mapsto \begin{cases} \{x\}_{n-1} \otimes x & : n \geq 3 \\ (1-x) \wedge x & : n = 2 \end{cases} \tag{33}$$

$$\delta_n : \qquad \{\infty\}, \{0\}, \{1\} \mapsto 0$$

The $\{x\}_n$ is the projection of $\{x\}$ in $\mathcal{B}_n(F)$. Set

$$\mathcal{A}_n(F) := \operatorname{Ker} \delta_n .$$

Any element

$$\alpha(t) = \Sigma n_i\{f_i(t)\} \in \mathbb{Z}[P^1_{F(t)}]$$

has a specialization

$$\alpha(t_0) := \Sigma n_i\{f_i(t_0)\} \in \mathbb{Z}[P^1_F], t_0 \in P^1_F.$$

(It is correctly defined even if t_0 is a pole of $f_i(t)$, in this case $f_i(t_0) = \infty \in P^1_F$).

Definition 13. *$\mathcal{R}_n(F)$ is generated by elements $\alpha(0) - \alpha(1)$ where $\alpha(t)$ runs over all elements of $\mathcal{A}_n(F(t))$, and also $\{\infty\}, \{0\}$.*

Lemma 14. $\delta_n(\mathcal{R}_n(F)) = 0$.

Proof. See proof of Lemma 1.16 in [G2].

So we get

$$\delta : \mathcal{B}(F) \to \begin{cases} \mathcal{B}_{n-1}(F) \otimes F^* & : n \geq 3 \\ \wedge^2 F^* & : n = 2 \end{cases}$$

Let me give some examples of elements of $\mathcal{R}_n(F)$.

Example 15. $\{x\}+\{x^{-1}\}$ and $\{x\}+\{1-x\} \in \mathcal{R}_2(F)$. Indeed, $\delta_2(\{x\}+\{x^{-1}\}) = (1-x) \wedge x + (1-x^{-1}) \wedge x^i = 0$ in $\wedge^2 F(t)^*$ modulo 2-torsion. On the other hand, $\{x\} + \{x^{-1}\}|_{x=\infty} \in \mathcal{R}_2(F)$ by definition. The same arguments work for $\{x\} + \{1-x\}$.

Example 16. $\{x\} + (-1)^n\{x^{-1}\} \in \mathcal{R}_n(F)$. Indeed, by induction $\delta_n(\{x\} + (-1)^n\{x^{-1}\}) = (\{x\} + (-1)^{n-1}\{x\}) \otimes x \in \mathcal{R}_{n-1}(F(t)) \otimes F(t)^*$ and $\{x\} + (-1)^n\{x^{-1}\}|_{x=\infty} \in \mathcal{R}_n(F)$ by definition. In particular, $2\cdot\{1\} \in \mathcal{R}_{2m}(F)$. (Put $x=1, n=2m$). We will prove below that $\{1\} \notin \mathcal{R}_{2m+1}(\mathbb{C})$.

Any real-valued function, and in particular $\mathcal{L}_n(z)$ (see (18)), defines a homomorphism

$$\tilde{\mathcal{L}}_n : \mathbb{Z}[\mathbb{C}P^1] \to R$$
$$\{z\} \mapsto \mathcal{L}_n(z)$$

Theorem 17 ([G4]). $\mathcal{L}_n(R_n(\mathbb{C})) = 0$.

Theorem 18. *Suppose that for some $f_i(t) \in \mathbb{C}(t)^*$ one has $\Sigma_i n_i \cdot \mathcal{L}(f_i(t)) = 0$. Then for any $z \in C$*

$$\sum_i n_i(\{f_i(z)\} - \{f_i(0)\} \in \mathcal{R}_n(\mathbb{C})\,.$$

So $\mathcal{R}_n(\mathbb{C})$ is the subgroup of all functional equations for n-logarithms. The canonical inclusion $R_2(F) \hookrightarrow \mathcal{R}_2(F)$ is an isomorphism. Indeed, the rigidity

$$K_3^{\text{ind}}(F) = K_3^{\text{ind}}(F(X))$$

(X is any irreducible curve over F) inplies that

$$H^1(B_F(2)) = H^1(B_{F(X)}(2))\,.$$

Therefore any functional equation for the dilogarithm $D_2(z)$ is a formal consequence of the 5-term functional equation.

Example 19. $\{1\} \notin \mathcal{R}_{2n+1}(\mathbb{C})$ because $\tilde{\mathcal{L}}_{2n+1}(1) = \zeta_{\mathbb{Q}}(2n+1) \neq 0$. There is the following complex $\Gamma_F(n)$:

$$\mathcal{B}_n \xrightarrow{\delta} \mathcal{B}_{n-1} \otimes F^* \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2 \otimes \wedge^2 F^* \xrightarrow{\delta} \wedge^n F^*$$

where $\mathcal{B}_n \equiv \mathcal{B}_n(F)$ is satisfied in degree 1 and

$$\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} \mapsto \delta(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i$$

has degree +1.

Conjecture 20. $H^i(\Gamma_F(n) \otimes \mathbb{Q}) \cong K_{2n-i}^{[n-i]}(F)$.

*Example 21.*Let $F = \mathbb{Q}$. We showed in Example 19 that $\{1\} \in \mathcal{R}_{2n+1}(\mathbb{Q})$. On the other hand $\delta\{1\} = 0$ by definition. So $\{1\}_{2n+1}$ should represent a nontrivial element in $\mathbb{K}_{4n+1}^{[2n]}(\mathbb{Q})$. Note that

$$\dim K_m(\mathbb{Q}) = \begin{cases} 1 & \text{for } m = 4n+1 \\ 0 & \text{otherwise}. \end{cases}$$

Complexes $\Gamma_F(n)$ should satisfy Beilinson–Lichtenbaum axioms, [B], [L]. In the case when F is a number field, Conjecture 20 essentially coincides with Zagier's conjecture about $K_{2n+1}(F)$. In this case (see [Y])

$$K_{2n+1}^{[m]}(F) = \begin{cases} K_{2n+1}(F) & \text{if } m = n \\ 0 & \text{otherwise}. \end{cases}$$

Note that by definition

$$H^n(\Gamma_F(n)) = K_n^M(F).$$

Conjecture 20 can be considered as a hypothetical "computation" of Quillen K-groups of an arbitrary field in terms of the same field.

12. Volumes of hyperbolic manifolds

According to the Gauss–Bonnet theorem, the volume of a compact even-dimensional hyperbolic manifold is proportional (with a universal constant c_n) to its Euler characteristic.

Conjecture 22. *Let X^{2n-1} be a $(2n-1)$-dimensional complete hyperbolic manifold of finite volume and curvature -1. Then there is an element*

$$\sum_i n_i\{z_i\} \in \mathbb{Q}\left[P^1_{\bar{\mathbb{Q}}}\right]$$

satisfying the condition (see (33))

$$\delta_n\left(\sum_i n_i\{z_i\}\right) := \sum_i n_i\{z_i\}_{n-1} \otimes z_i = 0$$

(respectively $\sum_i n_i(1-z_i) \wedge z_i = 0$ in $\wedge^2\bar{\mathbb{Q}}^*$) such that

$$vol(X^{2n-1}) = \sum n_i \mathcal{L}_n(z_i) \tag{34}$$

In the case $n = 2$ this follows immediately from results of [DS] or [NZ].

Theorem 23.[Go5] *Conjecture 22 is true for hyperbolic 5-manifolds.*

Let me sketch the proof for compact 5-manifolds. Note that

$$X^5 = H^5/\Gamma = B\Gamma$$

($B\Gamma$ is the classifying space of the discrete group Γ). The natural inclusion $\Gamma \hookrightarrow SO(5,1)$ induces a map

$$i : B\Gamma \to BSO(5,1)$$

Here $SO(5,1)$ is considered a discrete group. Recall that for a group G there is Milnor's simplicial model for BG:

$$x \leftleftarrows G \ \substack{\leftarrow\\ \leftarrow\\ \leftarrow} \ G^2 \ \substack{\leftarrow\\ \cdots\\ \leftarrow}$$

Let us denote by $I(g_0z, \ldots, g_5z)$ the geodesic simplex in the hyperbolic 5-space H^5 with vertices at points $g_0z, \ldots, g_5z$, where $g_i \in SO(5,1)$ and z is a given point in H^5. Now let us decompose X^5 on simplices

$$X^5 = \bigcup_i I(g_0^{(i)}z, \ldots, g_5^{(i)}z) \tag{35}$$

One can choose $g_j^{(i)}$ so that the boundary of the 5-chain

$$\sum_i (g_0^{(i)}, \ldots, g_5^{(i)}) \tag{36}$$

in $B\,SO(5,1)$ is 0 because of (35) and assumption $\partial X^5 = \phi$. On the other hand

$$\mathrm{vol}\,(I(g_0z, \ldots, g_5z))$$

is a continuous cocycle of $SO(5,1)$ representing a nonzero cohomology class of $H^5_{(m)}(SO(5,1), R)$ and hence a class $v_5 \in H^5(B\,SO(5,1), R)$. The value of v_5 on the cycle (36) is equal to vol (X^5) just by definition. Note that $H^{2n+1}_{(m)}(GL_N(\mathbb{C}), R)$ for a certain imbedding $SO(2n+1,1) \hookrightarrow GL_N(\mathbb{C})$. To complete the proof of Theorem 23 we need the following result proved in §3 of [G4]: there is a canonical homomorphism

$$f : H_5(GL_N(\mathbb{C})) \xrightarrow{f} H^1(B_{\mathbb{C}}(3))$$

such that the composition

$$H_5(GL_N(\mathbb{C})) \xrightarrow{f} H^1(B_{\mathbb{C}}(3)) \xrightarrow{\tilde{\mathcal{L}}_3} R$$

coincides with the Borel class in $H^5_{(m)}(GL_N(\mathbb{C}))$. This proves formula with $z_i \in \mathbb{C}$.

Proposition 24 (Ridgity). *Let*

$$z := \sum_i n_i\{z_i\} \in \mathbb{Z}[P^1_{\mathbb{C}}]$$

satisfies the condition $\delta_n(z) = 0$ *in* $\mathcal{R}_{n-1}(\mathbb{C}) \otimes \mathbb{C}^*$. *Then there is an element*

$$x := \sum_i n_i\{x_i\} \in \mathbb{Z}[P^1_{\bar{\mathbb{Q}}}]$$

such that $\delta_n(x) = 0$ *and* $\tilde{\mathcal{L}}_n(z) = \tilde{\mathcal{L}}_n(x)$.

Proof. Follows from the definition of the subgroup $\mathcal{R}_n(F)$ and Theorem 17.

It is interesting to compare Conjecture 22 with the following

Theorem 25. *The volume of a generic geodesic simplex in the Lobachevsky space* H^n *can* **not** *be expressed by the classical polylogarithms for* $n \geq 7$.

Volumes of geodesic symplexes in H^5 can be expressed by the trilogarithm ([Böhm], [Mu]). Volumes of geodesic simplexes in H^{2n} are expressible in terms of the lower dimensional spherical ones ([H]).

Conjecture 22 for compact manifolds can be deduced from conjecture 5.12 in [G4] using arguments analogous to the proof of Theorem 23.

References

[B1] Beilinson A.A., *Height pairings between algebraic cycles*, Lecture Notes in Math., (1289) p. 1–26.

[B2] Beilinson A.A., *Higher regulators and values of L-functions*, VINITI, **24** (1984), 181–238 (in Russian); English translation: J. Soviet Math., **30** (1985), 2036–2070.

[B-D] Beilinson A.A., Deligne P., *Polylogarithms and regulators,* To appear in Proceedings of the AMS Research Summer Conference "Motives."

[BMSch] Beilinson A.A., MacPherson R., Schechtman V.V., *Notes on motivic cohomology*, Duke Math. J., **54**(1987), 679–710.

[BGSV] Beilinson A.A., Goncharov A.B., Schechtman V.V., Varchenko A.N., *Projective geometry and algebraic K-theory*, Algebra and Analysis, (3) 1990, p. 78–131 (in Russian); Translated into English by A. M. S.

[B11] Bloch S., *Higher regulators, algebraic K-theory and zeta functions of elliptic curves*, Lect. Notes, U.C. Irvine, 1977. [B12] Bloch S., *Application of the dilogarithm function in algebraic K-theory and algebraic geometry,* Proc. Int. Symp. Alg. Geom., Kyoto (1977), 1–14.

[Böhm] J. Böhm, *Inhaltsmessung in R_5 konstanter Krümmung*, Arch. Math., (11)(1960), 298–309.

[Du1] Dupont J., *The Dilogarithm as a characteristic class for a flat bundle* , J. Pure and Appl. Algebra, **44**(1987), 137–164.

[DS] Dupont J., Sah C.-H., *Scissors congruencies II*, J. Pure and Appl. Algebra, **25**(1982), p. 159–195. Characteristic classes of flat bundles, Preprint 1992.

[GGL] Gabrielov A.M., Gelfand I.M., Losic M.V., *Combinatorial computation of characteristic classes*, Funct. Analysis and its Applications, **9**(2), (1975), p. 103–115, **3**(1975), p. 5–26, (in Russian).

[GM] Gelfand I.M., MacPherson R., *Geometry in Grassmanians and a generalization of the dilogarithm*, Advances in Math., **44**(1982), 279–312.

[G1] Goncharov A.B., *The classical trilogarithm, algebraic K-theory of fields, and Dedekind zeta-functions*, Bull. of the AMS, **29**(1), (1991), p. 155–161.

[G2] Goncharov A.B., *Geometry of configurations, polylogarithms and motivic cohomology*, Preprint of the Max-Planck-Institut für Mathematik, 1991. To appear in Advances in Mathematics.

[G3] Goncharov A.B., *Explicit construction of characteristic classes*, Preprint of the Max-Planck-Institut für Mathematik, 1992.

[G4] Goncharov A.B., *Polylogarithms and motivic Galois groups*, To appear in Proceedings of the AMS Research Summer Conference "Motives."

[Go5] Goncharov A.B., *Volumes of hyperbolic manifolds and mixed Tate motives*, Preprint 1993.

[H.-M] Hain R., MacPherson R., *Higher Logarithms*, Illinois J. of Mathematics, **34**(2), p. 392–475.

[H] Hopf H., *Die Curvatura integra Clifford–Kleinsher Raumformen*, Nachr. Acad. Wiss. Göttingen Math.-Phys. Kl., (1925), 131–141.

[K] Kummer E.E., *Über die Transzendenten, welche aus wiederholten Integrationen rationaler Formelen entstehen*, J. Reine Ang. Math., **21**(1840), pp. 74–90, 193–225, 328–371 (c.p. **11**(15)).

[L] Lewin L., *Dilogarithms and associated functions*, North Holland, 1981.

[Le] *Structural Properties of Polylogarithms*, L. Lewin, ed., AMS series of Mathematical Surveys and Monographs, 1991.

[Le] C.I. Gerhardt (ed), *G.W. Leibniz Mathematisch Schriften, II/1*, pp. 336–339, Georg Olms Verlag, Hildesheim. New York, 1971.

[L1] Lichtenbaum S., *Values of zeta functions at non-negative integers* , Journees Arithmetiques, Noordwykinhot, Netherlands, Springer Verlag, 1983.

[Ma] Matsumoto H., *Sur les sous groupes arithmétiques des groupes semisimples deployés*, Ann. Sci. Ec. Norm., Suppl(2)(1969), 1–62.

[Mo] Moore C., *Group extensions of p-adic and adelic linear groups*, Publ. Math. I.H.E.S., **35**(1969), 5–74.

[M] MacPherson R., *The combinatorial formula of Gabrielov, Gelfand and Losik for the first Pontryagin class*, Sem. Bourbaki 497, Fev. 1977, Sringer Lect. Notes in Math., 677.

[M2] Milnor J., *Algebraic K-theory and quadratic forms*, Inventiones Math., **9**(1970), 318–340.

[Mu] P. Müller, *Über Simplexinhalte in nichteuklidischen Räumen*, Dissertation Universität Bonn, 1954.

[NZ] W.D. Neumann, Zagier D.B., *Volumes of hyperbolic three-manifolds*, Topology, **24**(1985), 307–331.

[Q1] Quillen D., *Higher algebraic K-theory I*, Lect. Notes in Math., **341**(1973), 85–197.

[RS] Rapoport M., Schappacher N., Shneider M., *Beilinson's conjectures on values of L-functions*, Prog. in Math., 1989.

[S] Spence W., *An Essay on Logarithmic Transcendents*, pp. 26–34, London and Edinburgh, 1809.

[Sa] Sah C.-H., *Homology of classical groups made discrete, III*, J. of Pure and Appl. Algebra, (1989).

[So] Soulé C., *Operations en K-théorie Algébrique*, Canad. J. Math., **27**(1985), 488–550.

[S1] Suslin A.A., *Homology of GL_n, characteristic classes and Milnor's K-theory*, Proceedings of the Steklov Institute of Mathematics 1985, Issue 3, 207–226 and Springer Lecture Notes in Math., **1046**(1989), 357–375.

[S2] Suslin A.A., *Algebraic K-theory of fields*, Proceedings of the International Congress of Mathematicians, Berkeley, California, USA, 1986, 222–243. [S3] Suslin A.A., *K_3 of a field and Bloch's group*, Proceedings of the Steklov Institute of Mathematics (to appear).

[You] Youssin B.V., *Sur les formes $S^{p,q}$ apparaissant dans le calcul combinatoire de la deuxième classe de Pontrjaguine par la méthode de Gabrielov*, Gelfand, et Losik, C.R. Acad. Sci. Paris, Ser I, Math, **292**(1981), 641–649.

[Z1] Zagier D., *Polylogarithms, Dedekind zeta functions and the algebraic K-theorie of fields*, Arithmetic Algebraic Geometry (G.v.d. Geer F. Oort, J. Steenbrink, eds.), Prog. Math., **89**(1991), pp. 391–430, Birkhäuser, Boston.

[Z2] Zagier D., *Hyperbolic manifolds and special values of Dedekind zeta functions,* Inventiones Math., **83**(1986), 285–301.